TECHNOLOGY DEPT.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 65 NUMBER 9

CONTENTS

Linear Systems in Self-adjoint Form . . . . . C. LANCZOS 665
A Problem Arising in Yarn Spinning . . . . . STEPHEN KULIK 680
Comments on Lefschetz's Principle . . . . . A. SEIDENBERG 685
Mathematical Notes . . . HENRY L. ALDER, RICHARD BELLMAN 690
Classroom Notes. . . . . D. W. ROBINSON, M. E. LEVENSON, ... C. L. SEEBECK, JR. AND HENRY C. MILLER, JR. 694
Mathematical Education Notes . . . . P. C. ROSENBLOOM, ... F. A. FICKEN, JOHN D. BAUM 698
Elementary Problems and Solutions 708
Advanced Problems and Solutions 712
Recent Publications 719
News and Notices 722
The Mathematical Association of America 724
The Thirty-ninth Summer Meeting 724
Proposed Amendments to the By-Laws 727
Itineraries of Visiting Lecturers, 1958–59 728
New Members. 729
April Meeting of the Texas Section 731
Calendar of Future Meetings 734

NOVEMBER 1958

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

RALPH D. JAMES, *Editor*

ASSOCIATE EDITORS

RICHARD V. ANDREE	G. E. HAY	C. O. OAKLEY
JOHN A. BROWN	L. M. KELLY	J. M. H. OLMSTED
ROY DUBISCH	JOHN R. MAYOR	E. P. STARKE
HOWARD EVES	LLOYD J. MONTZINGO, JR.	H. S. ZUCKERMAN
	IVAN NIVEN	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. JAMES, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on 8½ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, University of Buffalo, Buffalo 14, N.Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, G. B. PRICE, University of Kansas
First Vice-President, G. B. THOMAS, JR., Massachusetts Institute of Technology
Second Vice-President, B. W. JONES, University of Colorado
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, LLOYD J. MONTZINGO, JR., University of Buffalo
Editor, R. D. JAMES, University of British Columbia

Additional Members of the Board of Governors: H. M. BACON, JEWELL H. BUSHEY, LAMBERTO CESARI, H. S. M. COXETER, P. H. DAUS, W. L. DUREN, JR., H. P. EVANS, HOWARD EVES, WILLIAM FELLER, N. J. FINE, J. S. FRAME, A. S. HOUSEHOLDER, G. B. HUFF, W. N. HUFF, G. K. KALISCH, E. C. KIEFER, J. C. KNIPP, W. G. LEAVITT, Z. L. LOFLIN, A. T. LONSETH, SAUNDERS MACLANE, J. R. MAYOR, E. J. MCSHANE, G. M. MERRIMAN, R. S. PARK, GEORGE PÓLYA, O. J. RAMLER, C. R. SHERER, M. F. SMILEY, R. G. SMITH, B. M. STEWART, F. M. STEWART, W. R. UTZ, JR., BERNARD VINOGRADE, CHARLES WEXLER, C. R. WYLIE, JR.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $5.00. For non-members the subscription price is $6.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y., during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Copyright © 1958 by The Mathematical Association of America, Inc.

PRINTED IN THE UNITED STATES OF AMERICA

LINEAR SYSTEMS IN SELF-ADJOINT FORM

C. LANCZOS, Dublin Institute for Advanced Studies

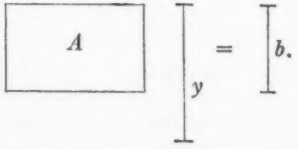
The principal axis problem of quadratic forms belongs to the best-investigated chapters of analysis. Much attention has been paid to the theory of matrices subjected to arbitrary linear transformations and the normal forms attainable by such transformations; (*cf.* [1],* p. 58). The relation of the general theory of linear equations to matrix calculus has also found exhaustive treatment; (*cf.* [2], Chapter IV). It seems, therefore, that the fundamental aspects of this field are practically exhausted.†

It is the purpose of the following discussions to approach the problem of general linear algebraic systems from a somewhat different viewpoint which throws new light on the nature of the principal axis problem by showing that the properties of symmetric matrices are extendable to arbitrary matrices to a surprisingly large degree, without demanding anything but *orthogonal* transformations. In this way the classical theory of linear algebraic forms, developed by Frobenius and Kronecker around the end of the last century (*cf.* [4], p. 268), which centers around the concept of the "rank" of a matrix, is elucidated from a totally different angle in which a certain *eigenvalue problem* plays the central role.

We formulate the given simultaneous system of n equations in m unknowns as the matrix equation

$$Ay = b, \tag{1}$$

in which A denotes an arbitrary n-row, m-column matrix—briefly denoted as an $n \times m$ matrix—while y is the unknown and b the given right side, both column vectors of m, respectively, n components. The matrix diagram associated with our system, (if we picture the case $n < m$), looks as follows:

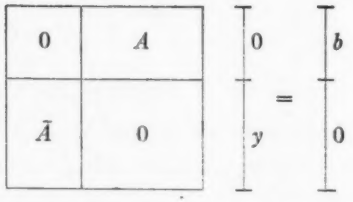


We will replace this diagram by the following extended diagram:‡

* Numbers in square brackets refer to the references at the end of the paper.

† A very extensive literature is digested in [3].

‡ The symbol "tilde" ($\sim$) refers to a transposition of rows and columns. If A has complex elements, the transposition shall include a change of i to $-i$ in every element. This generalization is so obvious that we will assume the reality of A and call S a "symmetric" rather than "Hermitian" matrix.



This diagram belongs to the equation $Sw=g$, where the symmetric square matrix $S=\bar{S}$ is defined as follows:

$$S = \begin{array}{|c|c|} \hline 0 & A \\ \hline \bar{A} & 0 \\ \hline \end{array}, \tag{3}$$

while the column vectors w and g have $n+m$ components, which can be displayed more conveniently by transposing them into row vectors:

$$\begin{aligned} \bar{w} &= \left|\begin{array}{c|c} (n) & (m) \\ \hline 0 & y \end{array}\right|, \\ \bar{g} &= \left|\begin{array}{c|c} & \\ \hline b & 0 \end{array}\right|. \end{aligned} \tag{4}$$

On the other hand, if these two vectors have the following structure:

$$\begin{aligned} \bar{w} &= \left|\begin{array}{c|c} (n) & (m) \\ \hline x & 0 \end{array}\right|, \\ \bar{g} &= \left|\begin{array}{c|c} & \\ \hline 0 & c \end{array}\right|, \end{aligned} \tag{5}$$

we obtain the equation

$$Ax = c, \tag{6}$$

and the number of equations is now *greater* than the number of unknowns. Hence we can without loss of generality assume that $n \leqq m$ and put

$$m = n + r \qquad (r \geqq 0). \tag{7}$$

(In our final results all reference to a preestablished relation between n and m will disappear and the numbers n and m be left completely arbitrary.) A general vector u of the $n+m$-dimensional space associated with the matrix S will be introduced as follows:

$$\tilde{u} = |\underbrace{\quad}_{x}\underbrace{\quad\quad}_{y}|. \tag{8}$$

Since S is symmetric, we can transform it into a diagonal matrix D with the help of an orthogonal transformation (the "principal-axis transformation"). The matrix U of this orthogonal transformation is composed of the $n+m$ column vectors u_i which satisfy the following eigenvalue problem:

$$Su_i = \lambda_i u_i \qquad (i = 1, 2, \cdots n+m) \tag{9}$$

with the added normalization

$$|u_i|^2 = 1. \tag{10}$$

As is well known, the eigenvalues λ_i of our problem are all *real.*

In view of the special form of the matrix S our eigenvalue problem separates into the two equations

$$Ay_i = \lambda_i x_i, \quad \tilde{A}x_i = \lambda_i y_i. \tag{11}$$

We will call this system of equations the "shifted eigenvalue problem" because on the right side the vectors x_i and y_i are in a "shifted" position, compared with the usual eigenvalue problem associated with A and $\tilde{A}$.

The eigenvalue problem (11) has a number of interesting properties which we are now going to demonstrate.

1. If we premultiply the second equation by A and substitute on the right side for Ay_i its value from the first equation, we obtain

$$A\tilde{A}x_i = \lambda_i^2 x_i. \tag{12}$$

Similarly, premultiplying the first equation by $\tilde{A}$ and making use of the second equation we obtain

$$\tilde{A}Ay_i = \lambda_i^2 y_i. \tag{13}$$

We thus see that the vectors x_i and y_i can be defined *separately* in themselves. They are the eigenvectors (principal axes) of the nonnegative symmetric matrices $A\tilde{A}$, respectively, $\tilde{A}A$. The first matrix is an $n\times n$, the second an $m\times m$ symmetric matrix. Hence the vectors x_i and y_i belong to two completely different spaces, the one of n, the other of m dimensions. The vectors x_i, put together columnwise, form an $n\times n$ complete orthogonal matrix X:

$$\tilde{X}X = X\tilde{X} = I. \tag{14}$$

The same can be done with the vectors y_j, resulting in the complete $m\times m$ orthogonal matrix $\overline{Y}$:

$$\tilde{\overline{Y}}\overline{Y} = \overline{Y}\tilde{\overline{Y}} = I. \tag{15}$$

2. The sequence of the columns of these two matrices is not arbitrary. The vectors x_i and y_i, although defined independently by the equations (12) and (13), are in actual fact *paired*, since they are also in the relation (11) to each other. To every y_i the corresponding x_i can be found by

$$x_i = \frac{1}{\lambda_i} A y_i \tag{16}$$

and vice versa, provided that λ_i is not zero. In fact, every solution of the system (11) with nonvanishing λ_i can immediately be extended to a *second* pair of vectors since a simultaneous change of λ_i to $-\lambda_i$ and y_i to $-y_i$ leaves the equations (11) unchanged. Accordingly, all nonvanishing eigenvalues will appear in *pairs* $\pm\lambda_i$ and we can agree that what we will call λ_i, shall be a *positive* number, complemented by the negative eigenvalue $-\lambda_i$. For the time being we want to assume that all the λ_i^2 associated with the eigenvalue problem (12) are different from zero. This provides us with the n positive numbers $\lambda_1, \cdots, \lambda_n$, complemented by the sequence $-\lambda_1, \cdots, -\lambda_n$ of negative eigenvalues.

3. An interesting property of the x_i, y_i vectors is that their length is automatically *equal*. We see from (16) that we get

$$\bar{x}_i x_i = \frac{1}{\lambda_i^2} \bar{y}_i \bar{A} A y_i = \bar{y}_i y_i. \tag{17}$$

For this reason, if the length of the vector x_i is normalized to 1, the length of the corresponding vector y_i becomes automatically 1.

4. The pairing of the vectors x_i, y_i cannot occur unlimitedly, since we cannot have more than n x-vectors while the number of y-vectors is m. Having obtained the $2n$ pairs (x_i, y_i) and $(x_i, -y_i)$, the remaining y_k-vectors can have no x_k-associates, which is only possible if $\lambda_k = 0$. Hence the eigenvalue zero is always present among the eigenvalues if r is not zero. Since the total number of eigenvalues is $n+m=2n+r$, we must have r eigenvectors associated with the eigenvalue zero. The corresponding x_k-vectors vanish. We consider these additional y-vectors as columns of a matrix Y_0 which has m rows and r columns. The complete $m \times m$ $\overline{Y}$-matrix is thus composed of the $m \times n$ Y-matrix, associated with the matrix X, and the additional $m \times r$ matrix Y_0, associated with the zero matrix. The complete U-matrix which contains all the principal axes of the matrix S, in the sequence $\lambda = -\lambda_1, \cdots, -\lambda_n; \lambda_1, \cdots, \lambda_n; 0, \cdots, 0$; appears in the form (18) while the diagonal matrix D into which S is transformed if we rotate it into the reference system of the principal axes, becomes (19). Here Λ denotes the $n \times n$ diagonal matrix whose diagonal elements are the n positive numbers $\lambda_1, \cdots, \lambda_n$.

Let us observe that the columns of the matrix U, being composed of the two

(18)
$$U = \begin{array}{c|ccc} & n & n & m-n \\ \hline n & X & X & 0 \\ \hline m & -Y & Y & \sqrt{2}Y_0 \end{array};$$

(19)
$$D = \begin{array}{c|cc|c} & n & & \\ \hline n & -\Lambda & & \\ \hline & n & \Lambda & m-n \\ \hline & & m-n & 0 \end{array}.$$

vectors x_i and y_i, both of the length 1, are normalized to $\sqrt{2}$ instead of 1. For this reason the columns of the matrix Y_0 have to be multiplied by $\sqrt{2}$, in order to uniformize all lengths of the columns of U. The right side of (10) has to be changed accordingly from 1 to 2.

We will now drop the restricting condition that all the λ_i of the eigenvalue problem (12) are positive. Generally only $p \leqq n$ eigenvalues need be different from zero. We thus introduce here a new number p which is characteristic for the matrix A, in addition to the two numbers n and m. This number, which in fact coincides with the "rank" of the matrix A, is here defined by the *number of independent eigensolutions of the system* (11) *which are possible if we demand that the eigenvalue* λ_i *shall be a positive number.* This p can take any value between 1 and the smaller of the two numbers (n, m):

(20)
$$1 \leqq p \leqq \min(n, m).$$

The case $p=0$ is excluded if A does not vanish identically because, if all eigenvalues of $A\bar{A}$ are zero, the whole matrix $A\bar{A}$ vanishes, which is only possible if $A=0$.

In view of this new number p the previous picture of the matrix U and the diagonal matrix D changes to some extent. The multiplicity of the eigenvalue zero has now increased from $m-n$ to $m+n-2p$. The matrix X, associated with the nonzero eigenvalues, is no longer an $n \times n$ but an $n \times p$ matrix. Moreover, the diagonal matrix Λ which appears in the construction of the matrix D, is no longer an $n \times n$ but a $p \times p$ matrix, composed of the positive numbers $\lambda_1, \cdots, \lambda_p$:

(21) $U =$

	p	p	$n-p$		$m-n$
n	X	X	X_0'	X_0'	0
m	$-Y$	Y	$-Y_0'$	Y_0'	$\sqrt{2}Y_0''$

,

(22) $D =$

	p	p	$m+n-2p$
p	$-\Lambda$		
p		Λ	
$m+n-2p$			0

.

We have thus obtained a detailed picture of the structure of the matrices U and D which are associated with the principal axis transformation of the matrix S. The entire principal axis problem is included in the matrix equation*

$$SU = UD \tag{23}$$

with the added condition

$$\bar{U}U = 2I. \tag{24}$$

If now we postmultiply (23) by $\bar{U}$, we obtain the relation

$$2S = UD\bar{U}. \tag{25}$$

We wish to construct S on the basis of this relation, performing the matrix multiplication indicated on the right side of (25). First we obtain the matrix

* For this formulation of the principal axis problem *cf.* [5], p. 93.

UD. We know that multiplication by a diagonal matrix as a second factor means that the columns of the first matrix are in succession multiplied by the successive diagonal elements of the second matrix. This gives

(26)
$$UD = \begin{array}{c|c|c|c} & p & p & n+m-2p \\ \hline n & -X\Lambda & X\Lambda & 0 \\ \hline m & Y\Lambda & Y\Lambda & 0 \end{array}.$$

Now we should transpose U and postmultiply by it. We can, however, leave U in its original form (21), if we agree that "row by column multiplication" is changed to "row by row multiplication." We will indicate this kind of multiplication by a little circle $\circ$. First of all we multiply the first n rows of (26) by the first n rows of (21). This gives $-X\Lambda \circ X+X\Lambda \circ X+0=0$. We continue by multiplying the first n rows of (26) by the last m rows of (21). This gives $X\Lambda \circ Y+X\Lambda \circ Y+0=2X\Lambda \circ Y=2X\Lambda\bar{Y}$. We now come to the product of the last m rows of (26) with the first n rows of (21): $Y\Lambda \circ X+Y\Lambda \circ X+0=2Y\Lambda \circ X=2Y\Lambda\bar{X}$. Finally the last m rows of (26) multiplied by the last m rows of (21) yield $-Y\Lambda \circ Y+Y\Lambda \circ Y+0=0$. The complete result is the following matrix:

(27)
$$2S = \begin{array}{c|c} 0 & 2X\Lambda\bar{Y} \\ \hline 2Y\Lambda\bar{X} & 0 \end{array}.$$

Comparison with the original form (3) of S gives the following fundamental result:

(28) $$A = X\Lambda\bar{Y},$$

(29) $$\bar{A} = Y\Lambda\bar{X}.$$

The second equation contains no new statement since it merely repeats the first relation in transposed form. The equation (28) contains the following fundamental

DECOMPOSITION THEOREM. *An arbitrary nonzero matrix can be written as the product of the $n\times p$ orthogonal matrix X, $(\bar{X}X=I)$, the $p\times p$ positive diagonal matrix Λ and the transpose of the $m\times p$ orthogonal matrix $Y(\bar{Y}Y=I)$.*

The remarkable fact about this theorem is that *the principal axes associated with the zero eigenvalue do not participate at all in the formation of the matrix A.* This has a profound effect on the solution problem of the equation $Ay=b$.

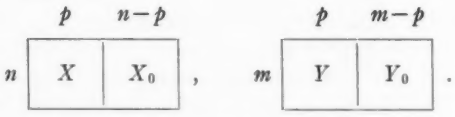
The matrices X, Y and Λ which appear in this theorem, are defined by the shifted eigenvalue problem

$$AY = X\Lambda, \qquad \tilde{A}X = Y\Lambda, \tag{30}$$

with the added condition that the diagonal elements of Λ shall all be nonzero positive numbers.

Example. Consider the $n\times m$ matrix whose elements are all zero, except the single element a_{ij} which may be given as the complex number c. Show that in this problem $p=1$, $\lambda_1=|c|$; all the n elements of the vector x_1 vanish except the element $x_1^{(0)}$; all the m elements of the vector y_1 vanish, except the single element $y_1^{(0)}$. Demonstrate the validity of the relation (28).

Before we continue with the further analysis of our problem, let us recall the two full $n\times n$ and $m\times m$ spaces associated with our eigenvalue problem. We can picture them as follows:



The matrices X and Y are composed of p mutually orthogonal axes of the two respective spaces. They form a p-dimensional subspace within the full space. Hence it is generally not permissible to convert the relation $\tilde{X}X=I$ to $X\tilde{X}=I$, and the same holds for the matrix Y. The remaining axes, included in the $n\times(n-p)$ orthogonal matrix X_0, respectively the $m\times(m-p)$ orthogonal matrix Y_0, belong to the eigenvalue zero and are thus defined by the two noninterrelated equations

$$\tilde{A}X_0 = 0, \tag{31}$$

respectively,

$$AY_0 = 0. \tag{32}$$

The columns of these two matrices are thus composed of the solutions of the homogeneous equation

$$\tilde{A}\overset{0}{x}_i = 0 \qquad (i = 1, \cdots, n-p), \tag{33}$$

respectively,

$$A\overset{0}{y}_j = 0 \qquad (j = 1, \cdots, m-p). \tag{34}$$

In harmony with the orthogonal nature of principal axes we assume that the

vectors x_i^0 are mutually orthogonal and their length is 1. The same can be said of the vectors y_j^0. This, however, is not self-evident since the principal axes belonging to a multiple eigenvalue (in this case the eigenvalue zero), are not orthogonal by nature, although they *can* be orthogonalized. We will have use for the vectors ξ_i, which are merely solutions of the homogeneous equation

$$\bar{A}\xi_i = 0 \qquad (i = 1, \cdots, n-p) \tag{35}$$

without demanding their orthogonalization and the normalization of their length. We merely demand that they shall be linearly independent and that their number shall be $n-p$, in order to span the entire space X_0. Similarly we will consider the $m-p$ linearly independent solutions of the homogeneous equation

$$A\eta_j = 0 \qquad (j = 1, \cdots, m-p) \tag{36}$$

without demanding their orthogonalization and normalization. The vectors ξ_i, taken as column vectors, form the $n\times(n-p)$ matrix Ξ_0, the vectors η_j the $m\times(m-p)$ matrix H_0. These matrices are no longer orthogonal but their orthogonality to the subspaces X and Y remains unchanged:

$$\tilde{\Xi}_0 X = 0, \qquad \tilde{H}_0 Y = 0. \tag{37}$$

With this preliminary information we return to the study of the linear equation (1) which could be done by diagonalization in the reference system of the principal axes of S, but we prefer to draw all our conclusions from the decomposition of our matrix A into the product (28). The equation (1) can now be written in the form

$$X\Lambda\bar{Y}y = b. \tag{38}$$

Premultiplication by the matrix $\tilde{\Xi}_0$ gives

$$0 = \tilde{\Xi}_0 b. \tag{39}$$

This equation, if written in the language of vectors, becomes

$$(\xi_i \cdot b) = 0 \qquad (i = 1, \cdots, n-p) \tag{40}$$

and we obtain the following (well-known)

COMPATIBILITY THEOREM. *The equation $Ay=b$ is solvable if and only if the given right side of the equation is orthogonal to every independent solution of the adjoint homogeneous equation $\bar{A}\xi=0$.*

That the condition (40) is *necessary*, follows from (39). That it is also *sufficient* follows from the fact that if b is perpendicular to the space Ξ_0 (or the equivalent space X_0), it must lie inside the space X, *i.e.* it must have the form

$$b = Xb'. \tag{41}$$

But in that case the equation (38)—premultiplying it by $\bar{X}$—gives at once

$$\Lambda \bar{Y} y = b'. \tag{42}$$

This equation is solvable by putting

$$y = Yy', \tag{43}$$

in which case we obtain

$$\Lambda y' = b', \; y' = \Lambda^{-1} b' \tag{44}$$

and finally

$$y = Y\Lambda^{-1}b' = Y\Lambda^{-1}\bar{X}b. \tag{45}$$

The diagonal matrix Λ contains only nonzero elements in the diagonal and is thus always invertible.

However, the solution (45) is not the *only* solution of our system. An arbitrary vector y, if analyzed in the reference system of the principal axes (Y, Y_0), appears in the following form:

$$y = Yy' + Y_0 y_0'.$$

If we put this expression in (38), we observe that the term with y_0' *drops out completely* from our equation. We thus obtain, as a counterpart of our previous Compatibility Theorem, the following

DEFICIENCY THEOREM: *The equation $Ay = b$ determines uniquely the projection of the vector y into the space Y but leaves its projection into the space Y_0 completely undetermined.*

We will now consider the solution of our system (1) under the following *auxiliary conditions*:

$$\bar{X}_0 b = 0 \quad \text{(by necessity)}, \tag{46}$$

$$\bar{Y}_0 y = 0 \quad \text{(by choice)}. \tag{47}$$

The second condition is not demanded by the original equation. By adding this condition we obtain a definite *particular solution* distinguished by the property that the solution finds its place in a *subspace of smallest capacity, viz.* the space Y.* This brings us back to the condition (43) and thus to the solution (45).

We know from the general theory of linear operators that the general solution of a linear system of equations is obtainable by adding to any particular solution an arbitrary solution of the homogeneous equation. Applying this principle to our problem we get

$$y = y_p + Y_0 \eta, \tag{48}$$

where η is an arbitrary column vector of $m-p$ elements while for y_p we can choose the particular solution (45), obtained under the auxiliary condition (47).

* This "normalization condition," which makes the solution unique, is equivalent to putting $y = \tilde{A}v$, where the vector v is unrestricted.

We can interpret the deficiency of the given system in the following terms. The equation (1) is not sufficient for the determination of y but it *may* become sufficient by *added information*. The information needed is a statement concerning the projection of y into the space Y_0. Hence we can conceive the equation (1) as *part of a more elaborate system*, the addition taking the form

$$\bar{Y}_0 y = b_0, \tag{49}$$

where b_0 is a free column vector of $m-p$ components. In this case the previous solution $y=y_p$ is not more than a *preliminary result*, while the complete solution takes the form (48). But now premultiplication by $\bar{Y}_0$ yields

$$\eta = b_0 \tag{50}$$

and the complete solution—obtained after complementing the original system (1) by the additional system (49)—becomes

$$y = y_p + Y_0 b_0 = Y\Lambda^{-1}\bar{X}b + Y_0 b_0. \tag{51}$$

The result of this analysis may be summarized as follows: The zero-fields X_0, Y_0, associated with the solutions of the homogeneous equations $\bar{A}x=0$, $Ay=0$, do not participate directly in the solution of the linear system (1) but merely decide the *compatibility* and the *deficiency* of the system. The compatibility conditions (46) have to be assumed in order to have a solution at all. The deficiency of the system can be removed by putting the solution in the space Y, assuming that the added information (49) will later provide the missing Y_0-portion of the solution. By this procedure an arbitrarily over-determined (although compatible) or under-determined (and thus deficient) linear system permits a unique solution.

Our solution can be put in the form

$$y = Gb, \tag{52}$$

where the $m\times n$ matrix G is defined as follows:

$$G = Y\Lambda^{-1}\bar{X}. \tag{53}$$

In view of the form (52) of the solution we can conceive the matrix G as the "inverse" of the matrix A. We should thus expect that the product GA has the property of the unit matrix I. It would be a mistake, however, to assume that the product GA must come out as the unit matrix. The product GA does not operate on an *arbitrary* vector b but on a vector which is subject to the condition (46). This means that b is inside the space X and has thus the form (41). Now

$$AG = X\Lambda\bar{Y}Y\Lambda^{-1}\bar{X} = X\bar{X} \tag{54}$$

and AG operating on b becomes

$$AGb = X\bar{X}b = X\bar{X}Xb' = Xb' = b, \tag{55}$$

which shows that AG has in fact the property of the unit matrix with respect to

all "permissible" vectors b.

On the other hand, let us premultiply (1) by G:

$$GAy = Gb = y. \tag{56}$$

This shows that the product GA must also have the property of the unit matrix, but again operating on a special class of vectors, subject to the condition (47). This condition puts y into the space Y which means that y can be put in the form (43). Now

$$GA = Y\Lambda^{-1}\bar{X}X\Lambda\bar{Y} = Y\bar{Y} \tag{57}$$

and therefore

$$GAy = Y\bar{Y}y = Y\bar{Y}Yy' = Yy' = y. \tag{58}$$

Once again the product GA has the property of the unit matrix I with respect to all permissible vectors y.

The matrix (53) has all the properties demanded by E. H. Moore in his "general analysis," (1906); (*cf.* [6]), establishing the "generalized inverse" of a matrix in abstract terms. We can likewise demonstrate that the conditions demanded by R. Penrose (*cf.* [7]; see also R. Rado, [8]) concerning the generalized inverse of a matrix are fulfilled. In our analysis the inverse matrix did not come about by any definitions in terms of matrix equations but by an *explicit method* of solving the linear system (1), based on the properties of an eigenvalue problem.

We can write out the solving matrix more explicitly by substituting for X and Y the constituting column vectors x_i and y_i which appeared in the solution of the eigenvalue problem (11). Let us denote the components of the p vectors x_α by $x_\alpha^{(i)}$ $(i=1, \cdots, n)$, the components of the conjugate vectors y_α by $y_\alpha^{(j)}$ $(j=1, \cdots, m)$. Then the element g_{ij} of the matrix G comes out as follows:

$$g_{ij} = \sum_{\alpha=1}^{p} \frac{y_\alpha^{(i)} x_\alpha^{(j)}}{\lambda_\alpha}, \tag{59}$$

while the element a_{ij} of the original matrix A becomes:

$$a_{ij} = \sum_{\alpha=1}^{p} x_\alpha^{(i)} \lambda_\alpha y_\alpha^{(j)}. \tag{60}$$

More important, however, is another interpretation of the solution (45). We know that the vector y lies inside the space Y which is composed of the p orthogonal vectors $y_1, \cdots, y_p$. Hence y can be analyzed in terms of these vectors:

$$y = \eta_1 y_1 + \eta_2 y_2 + \cdots + \eta_p y_p. \tag{61}$$

On the other hand, the right side b lies inside the space X and can be analyzed in terms of the conjugate orthogonal vectors $x_1, \cdots, x_p$:

$$b = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p \tag{62}$$

where the coefficients β_i are obtainable by projecting b on the axes X_i:

$$\beta_i = (b \cdot x_i). \tag{63}$$

We will call the two conjugate expansions (61) and (62) "co-orthogonal" since they involve two sets of orthogonal vectors which are in a one-to-one correspondence to each other.* Then the linear system (1), under the added auxiliary conditions (46) and (47), establishes the following relation between the coefficients β_i and η_i:

$$\eta_i = \beta_i/\lambda_i. \tag{64}$$

We see that the eigenvalues λ_i play the role of a "transfer function" in going from the right to the left, or from the left to the right.

Since the coefficients β_i are available on the basis of (63), the coefficients η_i become determined on the basis of (64) and the unknown vector y appears in the form of an orthogonal expansion (61), with given coefficients.

Operations in function space. The field of continuous linear operators—*i.e.*, the domain of linear differential or integral equations—can be handled on the basis of matrix operations if we introduce the infinite-dimensional "function space" and the matrices associated with this space; (*cf.* [9], p. 57). The results obtained in the theory of solving arbitrary linear algebraic systems can thus be extended to the realm of linear differential and integral equations. The characteristic feature of our investigation was that we have dealt with an arbitrarily over-determined or under-determined system and yet arrived at a unique solution under the proper auxiliary conditions.

The usual type of boundary-value problems considered in classical analysis are of the so-called "well-posed" type. This means that the given data—the differential equation with a given right side plus the boundary conditions—suffice for a unique solution and that the data can be prescribed freely, without the danger of incompatibility. Such problems realize in the language of matrices the case $n = m = p$: the number of equations is equal to the number of unknowns and the eigenvalue zero is not present (the matrix A is nonsingular).

Our investigation has shown that we can expect a valid and unique solution of a linear system under much more general conditions. The given operator (including the boundary conditions) may or may not comprise all the dimensions of the function space. The classical case usually considered belongs to those operators which comprise the *entire* function space, in both X and Y relations, *i.e.* in relation to the given right side as well as in relation to the unknown function. If we have a problem which is not "well-posed," this merely means that

* The expression "bi-orthogonal" would be misleading since it usually refers to two sets of *mutually orthogonal* vectors, while the two expansions (61) and (62) involve two sets of vectors which are orthogonal *within themselves*.

the given operator *omits* certain dimensions of the function space, either with respect to the given right side, or with respect to the unknown, or with respect to both. This, however, is no reason to reject the given boundary value problem. If the omission occurs with respect to the X-space, this means that the given data cannot be given freely but must be contained in a certain subspace of the function space. This condition can be met if we replace the word "prescribed data" by "observed data" because no matter how many surplus data we observe (in addition to the minimum number which would have sufficed for a unique solution), these data cannot be inconsistent since the prescribed mathematical law was in operation throughout our observations. Our operator is restricted to certain dimensions of the function space and cannot lead out of this space, no matter how many observations we perform.

If the given operator omits certain dimensions in the Y relation, this makes our solution incomplete since we obtain no information concerning the missing dimensions. We do get, however, a unique solution in those dimensions which are represented in the operator. We can then add later observations in order to complete our solution with respect to the missing dimensions.

We thus obtain a method for the solution of boundary value problems which can be arbitrarily over-determined (although consistent) or under-determined, and thus far from that "well-posed" type of problems that we expect under the customary conditions. Under these relaxed conditions the "inverse" of the operator does not exist any more in the ordinary sense. But even the "generalized inverse" in the sense of the matrix G which omits the zero-field and avoids the division by zero, need not necessarily exist. In the case of *finite* matrices it cannot happen that the matrix G, defined by (53) and more specifically by (59), should not exist. But in the case of continuous operators the corresponding expansion—called under simplified conditions the "bilinear expansion of the Green's function" (*cf.* [9], p. 360)—becomes an *infinite series* which may or may not converge. In many problems of an unconventional type to which the present theory is applicable, the bilinear expansion becomes in fact meaningless since it has no tendency to converge. Nor does the "inverse bilinear expansion" which corresponds to (60), converge and represent the given operator. This does *not*, however, interfere with the solution of our problem in terms of the two "co-orthogonal expansions" (61) and (62) which remain uniformly convergent even if the expansion of the inverse operator fails, provided that the right side is taken from that restricted subspace of the function space which is allotted to it by the nature of the given operator.

Although the application of the general theory to the field of continuous operators will be discussed in more detail in a separate paper*, it may not be without interest to give an example of the type of unconventional boundary value problems which become solvable by the method here presented.

* Proceedings of the International Congress of Mathematicians, Edinburgh, 1958 (to be published in 1960).

Problem. Given a simply-connected domain C of the complex plane, enclosed by a smooth boundary, let it be known that the function $f(z)$ of the complex variable $z=x+iy$ is analytical throughout C including the boundary, and let the value of $f(z)$ be given along the arbitrarily small arc S of the boundary. Find $f(z)$ inside the domain C.

The method of analytical continuation shows that this problem has a unique solution but the theory of analytical functions gives no clue toward a solution which would obtain the value of $f(z)$ at a distant point z directly in terms of the given boundary values along S.

The shifted eigenvalue problem associated with the present problem gives the solution in the following form. Associated with the domain C and the arc S we can define an infinite set of functions

$$f_1(z, z^*), \cdots, f_i(z, z^*), \cdots \tag{65}$$

which exist inside and on the boundary S, S', together with a corresponding set of functions defined along the arc S:

$$g_1(s), \cdots, g_i(s), \cdots. \tag{66}$$

The desired function $f(z)$ can be expanded into the infinite sum

$$f(z) = \sum_{\alpha=1}^{\infty} \gamma_\alpha f_\alpha(z, z^*) \tag{67}$$

where the expansion coefficients γ_i are obtained as follows:

$$\gamma_i = \int_S f(s) g_i(s) ds. \tag{68}$$

None of the functions $f_\alpha(z, z^*)$ are analytical (in view of the dependence on z^*, the "complex conjugate" of z). Nor do these $f_\alpha(z, z^*)$ satisfy the given boundary conditions. In fact, all the $f_\alpha(z, z^*)$ *vanish* along S. And yet, the infinite sum (67) *converges uniformly* to the correct value of $f(z)$ at every point of the domain C (including the outer boundary S') which excludes the arc S.

References

1. H. W. Turnbull and A. C. Aitken, Introduction to the Theory of Canonical Matrices, London, 1932.
2. R. A. Frazer, W. J. Duncan, and A. R. Collar, Elementary Matrices, Cambridge, 1950.
3. C. C. MacDuffee, The Theory of Matrices, New York, 1946.
4. Encyklop. der Math. Wissenschaften, vol. I, Leipzig, 1904.
5. C. Lanczos, Applied Analysis, Englewood Cliffs, N. J., 1956.
6. E. H. Moore, General Analysis, vol. 1, Amer. Philos. Soc., Philadelphia, 1935.
7. R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc., vol. 51, 1955, p. 406.
8. R. Rado, Note on generalized inverses of matrices, Proc. Cambridge Philos. Soc., vol. 52, 1956, p. 600.
9. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. 1, New York, 1953.

the given operator *omits* certain dimensions of the function space, either with respect to the given right side, or with respect to the unknown, or with respect to both. This, however, is no reason to reject the given boundary value problem. If the omission occurs with respect to the X-space, this means that the given data cannot be given freely but must be contained in a certain subspace of the function space. This condition can be met if we replace the word "prescribed data" by "observed data" because no matter how many surplus data we observe (in addition to the minimum number which would have sufficed for a unique solution), these data cannot be inconsistent since the prescribed mathematical law was in operation throughout our observations. Our operator is restricted to certain dimensions of the function space and cannot lead out of this space, no matter how many observations we perform.

If the given operator omits certain dimensions in the Y relation, this makes our solution incomplete since we obtain no information concerning the missing dimensions. We do get, however, a unique solution in those dimensions which are represented in the operator. We can then add later observations in order to complete our solution with respect to the missing dimensions.

We thus obtain a method for the solution of boundary value problems which can be arbitrarily over-determined (although consistent) or under-determined, and thus far from that "well-posed" type of problems that we expect under the customary conditions. Under these relaxed conditions the "inverse" of the operator does not exist any more in the ordinary sense. But even the "generalized inverse" in the sense of the matrix G which omits the zero-field and avoids the division by zero, need not necessarily exist. In the case of *finite* matrices it cannot happen that the matrix G, defined by (53) and more specifically by (59), should not exist. But in the case of continuous operators the corresponding expansion—called under simplified conditions the "bilinear expansion of the Green's function" (*cf.* [9], p. 360)—becomes an *infinite series* which may or may not converge. In many problems of an unconventional type to which the present theory is applicable, the bilinear expansion becomes in fact meaningless since it has no tendency to converge. Nor does the "inverse bilinear expansion" which corresponds to (60), converge and represent the given operator. This does *not*, however, interfere with the solution of our problem in terms of the two "co-orthogonal expansions" (61) and (62) which remain uniformly convergent even if the expansion of the inverse operator fails, provided that the right side is taken from that restricted subspace of the function space which is allotted to it by the nature of the given operator.

Although the application of the general theory to the field of continuous operators will be discussed in more detail in a separate paper*, it may not be without interest to give an example of the type of unconventional boundary value problems which become solvable by the method here presented.

* Proceedings of the International Congress of Mathematicians, Edinburgh, 1958 (to be published in 1960).

Problem. Given a simply-connected domain C of the complex plane, enclosed by a smooth boundary, let it be known that the function $f(z)$ of the complex variable $z=x+iy$ is analytical throughout C including the boundary, and let the value of $f(z)$ be given along the arbitrarily small arc S of the boundary. Find $f(z)$ inside the domain C.

The method of analytical continuation shows that this problem has a unique solution but the theory of analytical functions gives no clue toward a solution which would obtain the value of $f(z)$ at a distant point z directly in terms of the given boundary values along S.

The shifted eigenvalue problem associated with the present problem gives the solution in the following form. Associated with the domain C and the arc S we can define an infinite set of functions

$$f_1(z, z^*), \cdots, f_i(z, z^*), \cdots \tag{65}$$

which exist inside and on the boundary S, S', together with a corresponding set of functions defined along the arc S:

$$g_1(s), \cdots, g_i(s), \cdots. \tag{66}$$

The desired function $f(z)$ can be expanded into the infinite sum

$$f(z) = \sum_{\alpha=1}^{\infty} \gamma_\alpha f_\alpha(z, z^*) \tag{67}$$

where the expansion coefficients γ_i are obtained as follows:

$$\gamma_i = \int_S f(s) g_i(s) ds. \tag{68}$$

None of the functions $f_\alpha(z, z^*)$ are analytical (in view of the dependence on z^*, the "complex conjugate" of z). Nor do these $f_\alpha(z, z^*)$ satisfy the given boundary conditions. In fact, all the $f_\alpha(z, z^*)$ *vanish* along S. And yet, the infinite sum (67) *converges uniformly* to the correct value of $f(z)$ at every point of the domain C (including the outer boundary S') which excludes the arc S.

References

1. H. W. Turnbull and A. C. Aitken, Introduction to the Theory of Canonical Matrices, London, 1932.
2. R. A. Frazer, W. J. Duncan, and A. R. Collar, Elementary Matrices, Cambridge, 1950.
3. C. C. MacDuffee, The Theory of Matrices, New York, 1946.
4. Encyklop. der Math. Wissenschaften, vol. I, Leipzig, 1904.
5. C. Lanczos, Applied Analysis, Englewood Cliffs, N. J., 1956.
6. E. H. Moore, General Analysis, vol. 1, Amer. Philos. Soc., Philadelphia, 1935.
7. R. Penrose, A generalized inverse for matrices, Proc. Cambridge Philos. Soc., vol. 51, 1955, p. 406.
8. R. Rado, Note on generalized inverses of matrices, Proc. Cambridge Philos. Soc., vol. 52, 1956, p. 600.
9. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. 1, New York, 1953.

A PROBLEM ARISING IN YARN SPINNING

STEPHEN KULIK, University of South Carolina and Utah State University

1. Introduction. The problem discussed here arises in a cap spinning process. Thread passes through a small ring above the top of a cap (Fig. 1), slides over the edge of the cap, and winds on a bobbin which revolves on a spindle and moves up and down at the same time. The cap revolves with the bobbin but does not move up and down. Alternatively, a vertical motion of the cap instead of that by the bobbin can be assumed. In this way the layers of yarn are laid one over another. The manner in which the layers are arranged, and the final shape of the bobbin with yarn on it are of importance in maintaining a stable

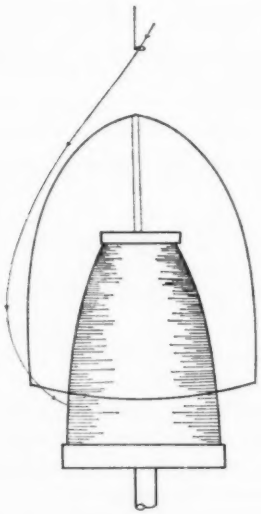


FIG. 1

shape when the bobbins are stored and during unwinding, particularly, when this operation is done by pulling the thread over the edge of the bobbin. The results of winding depend upon the shape of the barrel of the empty bobbin, the velocity of revolution and the velocity and amplitude of the vertical travel of the bobbin, and the rate of supply of yarn. The vertical travel of the bobbin is controlled in such a way that its speed may vary at different heights, but such that it passes each given height with the same speed regardless of which stroke it is making or whether it is an upward or downward stroke. Furthermore, the angular velocity of the bobbin is constant and the feed is arranged so that the rate of supply of yarn is also constant.

When the yarn is being built up on the bobbin, the radius of the barrel of

the bobbin increases and, therefore, the speed of sliding of the thread along the edge of the cap increases. Consequently, the twist of the yarn increases. There also may be a change of tension in the thread. We disregard the effect of these factors on the results of winding. Further, we assume that a unit of length of yarn occupies a fixed volume on the bobbin.

We consider two cases. In one case the bobbin travels a distance equal to the length of the bobbin each time. Equation (1) shows how the velocity of the motion of the bobbin can be determined in order to obtain a desired profile for the filled bobbin if we know the profile of the barrel of the empty bobbin. In the other case, considered in Section 3, the distance the bobbin travels is allowed to vary.

2. Constant length strokes. Let Figure 2 represent schematically half of an axial section of a bobbin turned on its side, where the axis of the bobbin coincides with the OX-axis, $y=f(x)$ is the equation of the barrel profile of the empty bobbin, and $y=F(x)$ is the equation of the required barrel profile of the filled bobbin, AD being the bottom and BC the top of the bobbin.

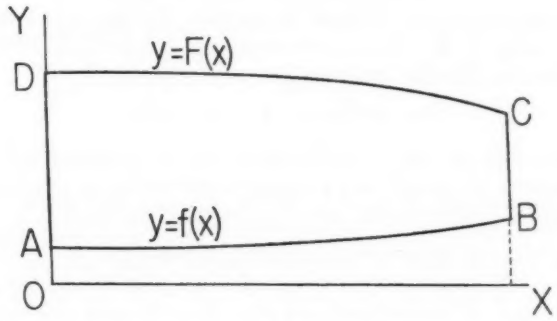


FIG. 2

Let the volume on the barrel filled with yarn in time t be $V=kt$, where k is a constant. Assume now for a moment that the bobbin is filled by one stroke from left to right. The velocity of the stroke is $v=dx/dt$, and we can write for the volume corresponding to dx, $dV=\pi[F^2(x)-f^2(x)]dx$. Therefore, the velocity as a function of the position is $v=(k/\pi)[F^2(x)-f^2(x)]$. The velocity should be increased n times if it is desired to have n layers of yarn on the bobbin. The forward strokes would then be followed by reverse strokes, the velocity of the reverse strokes having the opposite direction.

Hence, for the forward strokes, we have

$$v = (nk/\pi)[F^2(x) - f^2(x)]. \tag{1}$$

The thickness of layers is a function of x and the distance from the axis of

the bobbin. Starting from the empty barrel the thicknesses are given by

$$f_1(x) - f(x), f_2(x) - f_1(x), \cdots, F(x) - f_{n-1}(x), \tag{2}$$

where $y = f_m(x)$, $m = 1, \cdots, n-1$, is the equation of the profile of the bobbin partially filled by m strokes. The functions $f(x)$, $F(x)$, and $f_m(x)$ satisfy the equations

$$f_1^2(x) - f^2(x) = f_2^2(x) - f_1^2(x) = \cdots = F^2(x) - f_{n-1}^2(x). \tag{3}$$

It is easy to show that

$$f_m^2(x) = [mF^2(x) + (n - m)f^2(x)]/n \tag{4}$$

and

$$f_m^2(x) = [tF^2(x) + (T - t)f^2(x)]/T, \tag{4$_1$}$$

where t and T are the times required by m and n strokes respectively. If we write $f_0(x) = f(x)$, $f_n(x) = F(x)$, then (4) and (4$_1$) hold for $m = 0, 1, \cdots, n$.

In some problems it is convenient to consider the layers of yarn as infinitely thin, and the radius of an arbitrary cross-section of a partially-filled bobbin as a continuous function of t. In this case

$$y^2 = [tF^2(x) + (T - t)f^2(x)]/T \tag{5}$$

is the equation of the profile for any t satisfying the condition $0 \leqq t \leqq T$.

The equation of the profile of a partially-filled bobbin in terms of an ordinate y_1 at $x = x_1$, and of $f(x_1)$ and $F(x_1)$ can also be easily derived. It is

$$y^2 = [F^2(x) + k_1 f^2(x)]/(1 + k_1), \tag{6}$$

where

$$k_1 = [F^2(x_1) - y_1^2]/[y_1^2 - f^2(x_1)]. \tag{7}$$

3. A shortening device. A shortening device is introduced to regulate the length of strokes. Starting with the shortest, the length of strokes gradually increases to the full length of the bobbin. The bobbin still travels up and down with a velocity depending only on x as described in Section 2, except that the upward motion is reversed to downward before reaching its full length. This is useful when the final total thickness of the winding is to be made much less on the upper portion of the bobbin than on the lower. If only strokes of full length were used, the velocity of strokes would have to be very great toward the top of the bobbin. The rate v_e of the increase of the length of the strokes is derived in this section. We will call it the velocity of the shortening device, although we are not concerned with the mechanism of the shortening device itself.

Now let Figure 3 represent an axial section of the bobbin. The profile of the empty bobbin is ABD, that of the full bobbin is FE. The shortening device is

to start at B' and move to E'. We let $y=p(x)$ be the equation of BD. Also we let BC be an arbitrary curve, and we let $y=f(x)$ be the equation of ABC. The curve FE is given by the same equation as in Section 2, $y=F(x)$. In this way we define the velocity of strokes by equation (1), as if we were going to fill in a hypothetical bobbin with an empty profile ABC.

If BD is an arbitrary smooth curve, it is impossible to fill the volume on the bobbin between the points B and D in the required manner with a finite number of strokes. The actual profile between B and D would be, theoretically, a step line having common points with BD. We shall derive v_e for infinitely thin layers. In this way v_e is defined as a continuous function of x between B' and E'.

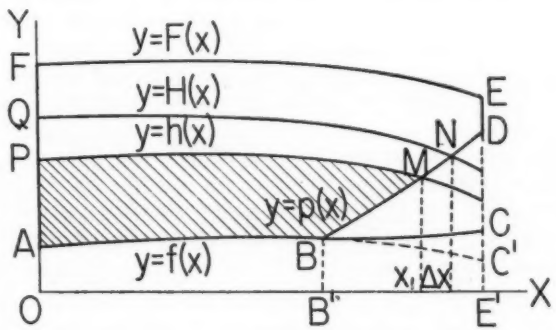


FIG. 3

Let the curves PM and QN be the profiles of the bobbin at times t and $t+\Delta t$ and their equations $y=h(x)$ and $y=H(x)$ respectively, where Δt is the time necessary to fill the volume $\Delta V=k\Delta t$ corresponding to the strip between these curves. Both curves PM and QN are extended hypothetically to the right of M and N by the definition of ABC and are determined by equation (6). Coordinates of the intersections of these curves with the curve given by the equation $y=p(x)$ are $(x_1,\ h(x_1))$ and $(x_1+\Delta x,\ H(x_1+\Delta x))$ respectively. In time Δt the length of strokes increases by Δx, and thus the velocity v_e at the point M can be written as $v_e=k \lim (\Delta x_1/\Delta V)$, as $\Delta x\to 0$. Now

$$\Delta V = \int_0^{x_1+\theta\Delta x} [H^2(x) - h^2(x)]dx \qquad 0<\theta<1.$$

Further,

$$h^2(x) = [F^2(x) + k_1 f^2(x)]/(1+k_1) \tag{8}$$

and

$$H^2(x) = [F^2(x) + k_2 f^2(x)]/(1+k_2), \tag{9}$$

where $k_1 = [F^2(x_1) - p^2(x_1)]/[p^2(x_1) - f^2(x_1)]$ and

$$k_2 = [F^2(x_1 + \Delta x) - p^2(x_1 + \Delta x)]/[p^2(x_1 + \Delta x) - f^2(x_1 + \Delta x)],$$

as it follows from (6) and (7).

Hence

$$H^2(x) - h^2(x) = \left(\frac{1}{1 + k_2} - \frac{1}{1 + k_1}\right)[F^2(x) - f^2(x)]. \tag{10}$$

Therefore,

$$\frac{dV}{dx} = \pi \frac{d}{dx}\{[p^2(x) - f^2(x)]/[F^2(x) - f^2(x)]\} \cdot \int_0^x [F^2(z) - f^2(x)]dz, \tag{11}$$

and finally

$$v_c = k \Big/ \left\{\pi \frac{d}{dx}[(p^2(x) - f^2(x))/(F^2(x) - f^2(x))] \cdot \int_0^x [F^2(z) - f^2(z)]dz\right\}, \tag{12}$$

where the subscripts on x_1 are omitted in both (11) and (12).

It should be remembered that the function $f(x)$ was chosen arbitrarily for x between B' and E'. It can be varied for a given bobbin and for a final profile. It will affect both v and v_c and presumably some choices would be preferred over others for practical reasons.

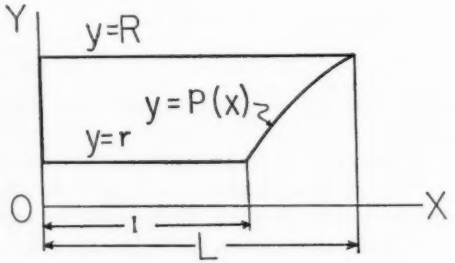


FIG. 4

Example. In Figure 4 let

$$F(x) = R, \quad 0 \leqq x \leqq L; \quad f(x) = r, \quad 0 \leqq x \leqq l.$$

It is required to find the equation of a curve $y = p(x)$ between the points (l, r) and (L, R) such that the velocity of the shortening device is constant.

The solution is $y = (k/v\pi) \log (x/l) + r^2$.

COMMENTS ON LEFSCHETZ'S PRINCIPLE

A. SEIDENBERG, University of California, Berkeley

The object of the present note is purely critical.

In his book [1], S. Lefschetz says: "In the present appendix we propose to show that, in a certain sense, algebraic geometry over a groundfield of characteristic zero may be reduced to complex algebraic geometry. This is without question the deep reason why characteristic zero algebraic geometry presents no new results over and above complex algebraic geometry."

Lefschetz goes on to explain that a variety V defined over the groundfield K (assumed algebraically closed) is actually determined by a finite number of quantities, namely, the coefficients in equations defining V. The field generated over the rational number field by these quantities is embedded isomorphically into the complex number field C and a complex variety V^* arises from the equations corresponding to those determining V. "The important fact," says Lefschetz, "is that in the passage from V to V^* all the strictly algebraic properties of V are preserved." But of course the question is whether results obtained by topological methods for the groundfield of the complexes, supposing their hypothesis and conclusion to be couched in algebraic terms, really are algebraic, that is, really are independent of the groundfield. Lefschetz thus merely assumes what he set out to prove.

Lefschetz discusses no typical example, but merely states three applications. The first two of these concern Picard's number ρ for surfaces. These theorems are not easy to understand, and one may wonder how appropriate they are for illustrating what is asserted to be a simple metamathematical principle. We'll come back to these examples but first consider a simpler one.

Consider two curves $f(x_0, x_1, x_2)=0$, $g(x_0, x_1, x_2)=0$ in the projective plane over the algebraically closed groundfield K of characteristic 0. In the complex case, it is known that the two curves meet in at least one point, that is, there exist numbers a_0, a_1, a_2 in C, not all $=0$, such that $f(a_0, a_1, a_2)=0$, $g(a_0, a_1, a_2)=0$. Presumably we are to conclude that the same is true for K. Let us follow the argument up and see. After the preparations indicated above, we may suppose K to be a subfield of C and to be the algebraic closure of a finite extension of the rational number field. We then conclude that the two curves have a point in common. Yes, but the meaning of the word *point* has shifted. The theorem we set out to prove was that there are triples of numbers (not all $=0$) in K which satisfy $f=0$ and $g=0$ and we have only concluded that there are such triples in an extension field of K.*

To be sure, the theorem of this example is a trifle, and one may suppose that all such trifles are to be taken for granted. In fact, it is undoubtedly true that some basic stock of theorems will have to be assumed in order to apply the

* One can scarcely suppose Lefschetz to be claiming only this weaker conclusion, as this would conflict with his assertion that "characteristic zero algebraic geometry presents no new results over and above complex algebraic geometry."

principle, and we may include the above. The point is, however, that Lefschetz makes the same sort of shift in his first illustrative example. In part this reads (p. 205):

THEOREM OF PICARD. *Corresponding to the algebraic surface* Φ *there exists a number* $\rho>0$, *the Picard number of* Φ, *such that*: (a) *there are sets of* ρ *irreducible curves* $C_1, \cdots, C_\rho$ *of* Φ *which cannot be the logarithmic curves of any closed differential of the third kind*; (b) *any* $\rho+1$ *irreducible curves of* Φ *are the logarithmic curves of some closed differential of the third kind of* Φ.

As in the first example, our comment here depends solely on the fact that the various objects in question—the surface Φ, the curves C_i, the differentials—are determined by coefficients from a given groundfield K (algebraically closed, of characteristic 0). Let us, then, see how the principle applies. After making the preparations indicated, we find curves $C_1, \cdots, C_\rho$, but the word *curve* has changed meaning: these curves are defined over an extension field of K, whereas what is being asserted, is that these curves are defined over K. That is what Lefschetz himself proved in Chapter IX of his book. The same objection holds for part (b): given $\rho+1$ curves of Φ (defined over K), the differential asserted to exist may have its coefficients in an extension field of K.

A. Weil considers ([4] p. 243) a metamathematical principle which we may call "Lefschetz's principle," since he himself calls it Lefschetz's principle, and which in other terms he expresses as follows: for given value of p, there is but one algebraic geometry of characteristic p. Considering especially the case $p=0$ he adds: "Lefschetz's principle is enough to show that the taboo against 'nonalgebraic,' *i.e.*, topological and function-theoretic methods, has little justification when only the case of characteristic 0 is to be considered."

While Weil does not justify this assertion by formal proof, he does try to vindicate it by considering a "typical example." This example uses all the leading ideas preceding the discussion, several which follow, together with a number of topological notions not dealt with in the book at all.* A perusal of Weil's considerations shows however that while we may regard "Lefschetz's principle" as established, the deeper principle which Lefschetz presumably intended has not been touched upon.

Weil obtains his conclusion not merely by changing the meaning of the words *point, curve, etc.*, but by changing the meaning of the words *algebraic geometry*. By his considerations applied to Picard's theorem, one concludes no more than that the curves $C_1, \cdots, C_\rho$ are defined over an extension field of K. To be sure, that is all that Weil would claim (though it would be easy to misunderstand the remark quoted above). But while Weil is certainly free to define algebraic geometry as he desires—and this definition is admittedly a creative act—all the same it does not change the algebraic facts, relative to Picard's theorem say, or even

* The "typical example" asserts the existence and uniqueness of certain rational numbers c_γ. Weil shows their existence but omits establishing the uniqueness.

the assertion that Picard's theorem, in the stricter sense we're taking it, that is, relative to an algebraically closed base field K, belongs to algebraic geometry. The rational number field is a distinguished field, as is its algebraic closure, and if we take K to be this algebraic closure, then Picard's theorem relative to K belongs to algebraic geometry in the sense of Weil. Moreover it is a true theorem, but its truth does not follow from "Lefschetz's principle."

"Lefschetz's principle," that is, the assertion that there is but one algebraic geometry of characteristic 0 in the sense of Weil, is of relatively limited value, and therefore should perhaps not be given Lefschetz's name, since it is clear that Lefschetz had more than this in mind, even though nothing else (by way of proof) can be gathered from his remarks.

We will now indicate how Picard's theorem, in the strict sense, follows from its truth over the complex numbers. The argument uses only metamathematical notions, applied to known results over the complex number field, plus a few basic, and simple, facts from algebraic geometry.* By algebraic geometry we shall mean algebraic geometry in the sense of Zariski, van der Waerden, and others; that is, relative to an arbitrary base field, except that for present purposes the base fields are restricted to be algebraically closed.†

Generalizing a phenomenon occurring in our first example, one can say that if a finite system of equalities and inequalities with coefficients in an algebraically closed field K has a solution in an extension field of K, then it already has a solution in K. Essentially the reason for this is that the elimination of the unknowns from the statement that the system has a solution in an algebraically closed extension field L of K proceeds uniformly for all L.

To make a still further convenient generalization we define the notion of *elementary sentence*, relative to a base field K. By an *atomic formula* we mean an expression of the form $f=0$, where f is a polynomial with coefficients in K. By a *formula* we mean an expression built up in a finite number of steps from atomic formulae by means of *conjunction*, *negation*, and *quantifiers of the form* $E(x)$, "there exists an x such that," where x varies over an algebraically closed field L containing K. An *elementary sentence* is a formula involving no free variables (parameters). Then, because the elimination of the variables proceeds uniformly, one obtains the following:

MINOR PRINCIPLE OF LEFSCHETZ.§ *If an elementary sentence holds for one algebraically closed field (containing K), then it holds for every algebraically closed field (containing K).*

* All these basic facts can be found, for example, in [3]. No doubt even these could be eliminated, but the result would be to encumber the discussion. More important, one couldn't get it into a small space.

† Thus Lüroth's theorem, for example, since it employs an arbitrary base field, is excluded from our body of theorems, although the same theorem for an algebraically closed base field is retained.

§ This name for the theorem is convenient in context, but for historical reasons might more correctly be given as the Principle of Tarski (for algebraically closed fields); see [2].

If every theorem of algebraic geometry (relative to an algebraically closed base field) were an elementary sentence, then Lefschetz's principle would be hereby established, but it is easy to give theorems which are not elementary sentences, for example, Hilbert's Nullstellensatz,* which says that if the system $F_1=0, \cdots, F_s=0, G\neq 0$, where $F_1, \cdots, G$ are polynomials in n letters X_i over the algebraically closed field K, has no solution in K, then $G^t=A_1F_1+\cdots+A_sF_s$ for some t and some polynomials $A_1, \cdots, A_s$ in $K[X_1, \cdots, X_n]$. For a given integer h, the statement that $G, G^2, \cdots$, or G^h is of the form $A_1F_1+\cdots+A_sF_s$ for some polynomials $A_1, \cdots, A_s$ of degree at most h, is, indeed, elementary (in the coefficients of the A_i), but as an *a priori* limit on h is not built into the theorem, the theorem is not elementary. On the other hand, Hilbert's theorem is the disjunction of these elementary sentences for $h=1, 2, \cdots$, and hence one sees that if the theorem holds for one algebraically closed field (containing K), then it holds for any other. It will be convenient to call Hilbert's theorem an almost-elementary sentence, where by an *almost-elementary* sentence we mean a sentence to which the Minor Principle applies with minor adjustments. Also if every theorem of algebraic geometry were almost-elementary, one could consider Lefschetz's principle as established, and our criticism as largely cancelled.

Coming to Picard's theorem, we first consider part (b). Let Φ be defined over K; and let K_1, a subfield of K, be the algebraic closure of a finite extension of the rationals over which Φ is still defined. Let K_1 be mapped by an isomorphism into C and identify K_1 with its image. We take $\rho=\rho(\Phi^*)$, where Φ^* is the complex surface corresponding to Φ. There may be curves on Φ not defined over K, but we assume, as does Lefschetz, that the given curves $C_1, \cdots, C_{\rho+1}$ are defined over K; and then, moreover, we may suppose them defined over K_1. Let $x_1, \cdots, x_n$ be a general point† of Φ/K_1 and let x_1, x_2 be algebraically independent over K_1. Let

$$\omega_1 = \sum P_i(x_1, \cdots, x_n)dx_i/Q(x_1, \cdots, x_n),$$

with $i=1, 2$, be a closed differential of the third kind having $C_1, \cdots, C_{\rho+1}$ as logarithmic curves.§ Here P_1, P_2, Q are polynomials over C, and if P_1, P_2, Q are of degree at most h we will say that ω_1 is of degree at most h; let ω_1 be of degree at most h. The statement that ω_1 is closed is (aside from the condition $Q\neq 0$) a conjunction of polynomial equality conditions on the coefficients (of general P_1, P_2, Q of degree at most h); the statement that at least one of $C_1, \cdots, C_{\rho+1}$ is a logarithmic curve is similarly a disjunction of polynomial inequalities. There is still the condition that no other curve is logarithmic. If all

* Though the variants of a theorem may for many purposes be regarded as equivalent, in a metamathematical argument it is necessary to make explicit the form intended.

† The argument is thus being referred to a definite affine space, but the events at infinity give no extra difficulties.

§ According to the terminology of Lefschetz, this means that at least one of $C_1, \cdots, C_{\rho+1}$ is a logarithmic curve and that there is no logarithmic curve other than these.

curves came into consideration, then one might run into difficulties, but only the curves cut out by $Q=0$ can be logarithmic curves (at finite distance). Hence one sees without difficulty that the additional assertions are polynomial equalities and inequalities on the coefficients. Since this system of polynomial equalities and inequalities does have a solution, it also has a solution in K_1, whence the existence of the desired differential follows.

Otherwise put, part (b) is the conjunction of disjunctions of elementary sentences, and thus is almost-elementary. That is not the case with part (a), however, at least if we take $\rho=\rho(\Phi^*)$ as in part (b). Here, let $C_1, \cdots, C_\rho$ be curves given by the theorem over C. Let h_i be the order of C_i (equal to the number of intersections with general hyperplane) and let $h=\max\{h_i\}$. Then we can insert h into part (a) and state that there are curves $C_1, \cdots, C_\rho$ of order at most h which are not the logarithmic curves of any closed differential ω_1 of the third kind. The statement that there is no such ω_1 of order at most a given k then would come to an elementary sentence, because it is known how to look upon the curves on Φ^* of order at most h as the points of a certain (possibly reducible) algebraic variety over K_1.* For each k we could get curves $C_1^{(k)}, \cdots, C_\rho^{(k)}$ defined over K_1, but it is not clear from this argument that one could select the $C_1^{(k)}, \cdots, C_\rho^{(k)}$ to be independent of k. This is the difficulty.

One could take the minimum ρ of (b) to be the ρ of (a),† so that again by an appropriate interpretation it can be claimed that (a) follows directly from Lefschetz's principle, although then the real problem is to see that ρ does not change upon extension of the base field. In any event, the point remains that there are theorems in algebraic geometry which are not almost elementary.

To get part (a), with $\rho=\rho(\Phi^*)$, we will make use of Severi's theory of the base, but in what follows it will not be necessary to have the definition of the base, but only a few of its properties which we shall mention as we go along. First, a base (for the curves on Φ^*) cosists of ρ curves $C_1, \cdots, C_\rho$, where ρ is the Picard number. Let C_i be of order h_i. As mentioned, the curves on Φ^* of order h_i constitute an algebraic variety V over K_1. Let C_i be given by a point P_i on the component V_i.§ Then, as far as the use which Severi makes of these curves, the point P_i may be replaced by any other point P_i' on V_i, provided P_i' represents an irreducible curve. As a consequence, we may take the curves C_i to be defined over K_1. Next, we make use of the fact, which can easily be read off from Severi's explicit results, that the curves of a base cannot be the logarithmic curves of any closed differential of the third kind. Combining these points, we get Picard's theorem for K_1. An additional slight argument gives the theorem over K.

Underlying Lefschetz's principle is not only the notion of *isomorphism*,

* The points of this variety represent the curves both reducible and irreducible, and it will be necessary to avoid those which represent reducible curves; these, however, fill out a variety over K_1.

† The inequality $\rho \geqq 1$ follows from the same inequality over the complex field.

§ The components of V over C are the same as those over K_1.

which both Lefschetz and Weil mention, but also the notion of *computability*, which neither mention. Partly, though perhaps not mainly, on the basis of the above considerations we make the following two conjectures.

CONJECTURE 1. *Lefschetz's principle is a valid principle.*

CONJECTURE 2. *If all of the results of algebraic geometry are taken into account, then each of them can, without difficulty, be strengthened to an almost elementary sentence.*

The establishment of Conjecture 2 would be an interesting but thankless task; interesting, because it would require an encyclopaedic perusal of the literature; thankless, because once done, one would see that it could, by definition, be done without difficulty.

References

1. S. Lefschetz, Algebraic Geometry, Princeton. 1953.
2. A. Tarski, A decision method for elementary algebra and geometry, (Prepared for publication by J. C. C. McKinsey) Berkeley, 1951.
3. B. L. van der Waerden, Einführung in die Algebraische Geometrie, New York, 1945.
4. A. Weil, Foundations of Algebraic Geometry, Amer. Math. Soc. Coll. Pub., vol. 29, New York, 1946.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California

A GENERALIZATION OF THE EULER ϕ-FUNCTION

HENRY L. ALDER, University of California, Davis

The Euler ϕ-function can be defined as the number of ordered pairs $\langle x, y\rangle$ for which $x+y=n$, $1\leqq x\leqq n$, and x and y both relatively prime to n. That this definition is equivalent to the usual one is immediately evident by observing that from $x+y=n$ and $(x, n)=1$, it follows that $(y, n)=1$. A natural generalization of this definition of the Euler ϕ-function, suggested by S. K. Stein, is the following:

DEFINITION. *The function $\phi(n, m)$ with $m\geqq 0$ is defined as the number of ordered pairs $\langle x, y\rangle$ for which $x+y=n+m$, $1\leqq x\leqq n$, and x and y both relatively prime to n.*

THEOREM. *$\phi(n, m)$ is given as follows: Let $n=p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s}$ be the prime power decomposition of n, then*

$$\phi(n, m) = n \prod_{r=1}^{s} \left(1 - \frac{\epsilon_m(p_r)}{p_r}\right), \tag{1}$$

where $\epsilon_m(p) = 1$ *if* $p \mid m$, $\epsilon_m(p) = 2$ *if* $p \nmid m$.

The well-known formula for the Euler ϕ-function

$$\phi(n) = \phi(n, 0) = n \prod_{r=1}^{s} \left(1 - \frac{1}{p_r}\right) \tag{2}$$

follows immediately from the above general formula, since in this case every p_r divides $m = 0$.

For the special case where $m = 1$, we obtain

$$\phi(n, 1) = n \prod_{r=1}^{s} \left(1 - \frac{2}{p_r}\right), \tag{3}$$

as in this case none of the p_r divides $m = 1$. Since $\phi(n, 1)$ is also equal to the number of pairs of consecutive positive integers each less than n and relatively prime to n, this formula is identical to the one by V. Schemmel [1].

It can immediately be seen that the same formula as in (3) would always result for $\phi(n, m)$, whenever m is relatively prime to n. This special case is equivalent to a result obtained by D. N. Lehmer ([2] p. 538) for the number of pairs of positive integers, each relatively prime to n, having a fixed difference λ relatively prime to n with the first of the pair being less than n.

Proof of theorem. We shall show first that $\phi(n, m)$ is multiplicative, *i.e.* if $(a, b) = 1$, then $\phi(ab, m) = \phi(a, m)\phi(b, m)$.

To prove this, let

$$S_1 = \{x_1, \cdots, x_{\phi(a,m)}\}$$

be the set of all values of x for which $x + y = a + m$, $1 \leqq x \leqq a$, and x and y both relatively prime to a. Now, if $(a, b) = 1$, then it can be shown that for each x for which $x + y = ab + m$, $1 \leqq x \leqq ab$, and x and y both relatively prime to ab, we must have

$$x \equiv x_i \pmod{a}, \tag{4}$$

where $x_i \in S_1$. For, suppose $x \equiv x_j \pmod{a}$, $1 \leqq x_j \leqq a$, where x_j is not an element of S_1, then $x = x_j + ka$ and for some y

$$x_j + ka + y = ab + m, \tag{5}$$

and for some z

$$x_j + z = a + m, \tag{6}$$

where either $(x_j, a) = e_1 > 1$ or $(z, a) = e_2 > 1$, since otherwise $x_j \in S_1$. Now $(x_j, a) = e_1$ implies that $e_1 \mid x$ and $e_1 \mid (x, ab) = 1$, so that $(z, a) = e_2 > 1$. But subtracting (6) from (5) shows that $e_2 \mid (y, ab) = 1$, which proves (4).

Therefore, if $x+y=ab+m$, $1 \leqq x \leqq ab$, and x and y both relatively prime to ab, then x is of one of the forms

$$x_i,\ x_i + a,\ x_i + 2a,\ \cdots,\ x_i + (b-1)a, \tag{7}$$

where $x_i \in S_1$. For each i, the set of numbers given by (7) constitutes a complete residue system, modulo b, since $(a, b)=1$. Now let

$$S_2 = \{x'_1, \cdots, x'_{\phi(b,m)}\}$$

be the set of all values of x for which $x+y=b+m$, $1 \leqq x \leqq b$, and x and y both relatively prime to b. Then repeating the argument used to derive (4), we observe that only those elements of (7) can satisfy the given conditions, *i.e.*, $x+y=ab+m$, $1 \leqq x \leqq ab$, $(x, ab)=1$, $(y, ab)=1$, which are congruent to some element of S_2, modulo b. Thus, for each i, there are $\phi(b, m)$ values in the set given by (7) which are easily seen to satisfy all the required conditions so that, since there are $\phi(a, m)$ values of i, we have $\phi(ab, m)=\phi(a, m)\phi(b, m)$.

Next, we shall show that

$$\phi(p^\alpha, m) = p^\alpha\left(1 - \frac{2}{p}\right), \qquad \text{if } p \nmid m. \tag{8}$$

As x runs through the numbers $1, \cdots, p^\alpha$, there are $p^{\alpha-1}$ values of x which are divisible by p and consequently have to be excluded to satisfy the required conditions. Since $x+y=p^\alpha+m$, it is impossible that both x and y are divisible by p, as $p \nmid m$, and, since y runs through a complete residue system, mod p^α, as x runs through the numbers $1, \cdots, p^\alpha$, there are $p^{\alpha-1}$ values of y which are divisible by p, which occur in ordered pairs $\langle x, y\rangle$ distinct from those for which x is divisible by p. We, therefore, have

$$\phi(p^\alpha, m) = p^\alpha - 2p^{\alpha-1} = p^\alpha\left(1 - \frac{2}{p}\right).$$

To show that

$$\phi(p^\alpha, m) = p^\alpha\left(1 - \frac{1}{p}\right) \qquad \text{if } p \mid m,$$

we merely observe that in this case the ordered pairs $\langle x, y\rangle$ for which $x+y=p^\alpha+m$ in which x is divisible by p are identical with those for which y is divisible by p, so that

$$\phi(p^\alpha, m) = p^\alpha - p^{\alpha-1} = p^\alpha\left(1 - \frac{1}{p}\right).$$

References

1. V. Schemmel, Ueber relative Primzahlen, J. Math., vol. 70, 1869, pp. 191–192.

2. D. N. Lehmer, On the congruences connected with certain magic squares, Trans. Amer. Math. Soc., vol. 31, 1929, pp. 529–551.

NOTES ON MATRIX THEORY—XV
MULTIPLICATIVE INEQUALITIES OBTAINED FROM ADDITIVE INEQUALITIES

RICHARD BELLMAN, RAND Corporation

1. Introduction. Let A be a positive definite matrix of order N, and let λ_i, $i=1, \cdots, N$, be its characteristic roots arranged in decreasing order of magnitude, $\lambda_1 \geqq \cdots \geqq \lambda_N$. As was shown by Ky Fan (see, for example, [2]) we have the following inequalities

$$\begin{aligned} &\prod_{i=k}^{N} \lambda_i \leqq \prod_{i=k}^{N} (x^i, Ax^i), \\ &\sum_{i=k}^{N} \lambda_i \leqq \sum_{i=k}^{N} (x^i, Ax^i), \qquad k = 1, \cdots, N \end{aligned} \tag{1}$$

for any set of orthonormal vectors $\{x^i\}$.

In a previous note [1] we showed how to obtain the additive inequality from the multiplicative one. In this note, we shall show that the multiplicative inequality can be obtained from the additive one.

2. Preliminary result. Consider the sum

$$\begin{aligned} c_k\lambda_k + c_{k+1}\lambda_{k+1} + \cdots + c_N\lambda_N &= c_k(\lambda_k + \lambda_{k+1} + \cdots + \lambda_N) \\ &\quad + (c_{k+1} - c_k)(\lambda_{k+1} + \cdots + \lambda_N) + \cdots + (c_N - c_{N-1})\lambda_N \end{aligned} \tag{1}$$

(Abel summation by parts).

If we assume that $0 \leqq c_k \leqq c_{k+1} \leqq \cdots \leqq c_N$, we have, upon using the additive inequality in (1.1), the result that

$$\begin{aligned} c_k\lambda_k + c_{k+1}\lambda_{k+1} + \cdots + c_N\lambda_N &\leqq c_k\left(\sum_{i=k}^{N} (x^i, Ax^i)\right) \\ &\quad + (c_{k+1} - c_k)\left(\sum_{i=k+1}^{N} (x^i, Ax^i)\right) + \cdots + (c_N - c_{N-1})(x^N, Ax^N) \\ &\leqq c_k(x^k, Ax^k) + c_{k+1}(x^{k+1}, Ax^{k+1}) + \cdots + c_N(c^N, Ax^N), \end{aligned} \tag{2}$$

for any set of orthonormal vectors $\{x^i\}$.

3. Multiplicative inequality. From (2.2) we deduce that

$$\min_R \left(\sum_{i=k}^{N} c_i\lambda_i\right) \leqq \min_R \left(\sum_{i=k}^{N} c_i(x^i, Ax^i)\right), \tag{1}$$

where R is the region in c-space defined by the relations

$$\begin{aligned} &\text{a.} \quad 0 < c_k \leqq c_{k+1} \leqq \cdots \leqq c_N, \\ &\text{b.} \quad \prod_{i=k}^{N} c_i = 1. \end{aligned} \tag{2}$$

Since the arithmetic-geometric mean inequality

$$(3) \qquad \sum_{i=k}^{N} c_i\lambda_i/(N-k+1) \geqq \left(\prod_{i=k}^{N} c_i\right)^{1/(N-k+1)} \left(\prod_{i=k}^{N} \lambda_i\right)^{1/(N-k+1)}$$

shows that the minimum of $\sum_{i=k}^{N} c_i\lambda_i$ is assumed at the points $c_i = a/(\lambda_i)$, where $a = (\prod_{i=k}^{N} \lambda_i)^{1/(N-k+1)}$, we see that the left-hand side of (1) is $(\prod_{i=k}^{N} c_i)^{1/(N-k+1)}$.

Let us now temporarily restrict our attention to orthonormal sets satisfying the additional restriction

$$(4) \qquad (x^k, Ax^k) \leqq (x^{k+1}, Ax^{k+1}) \leqq \cdots \leqq (x^N, Ax^N).$$

Then the right-hand side of (1) is $(\prod_{i=k}^{N} (x^i, Ax^i))^{1/(N-k+1)}$.

Hence we have the inequality

$$(5) \qquad \prod_{i=k}^{N} \lambda_i \leqq \prod_{i=k}^{N} (x^i, Ax^i)$$

for all orthonormal sets satisfying (4). From the symmetry of the condition, we see that this is no essential restriction. Hence the inequality holds for all orthonormal sets.

References

1. R. Bellman, Notes on matrix theory—VII, this MONTHLY, vol. 62, 1955, pp. 647–648.

2. Ky Fan, On a theorem of Weyl concerning eigenvalues of linear transformations, I, Proc. Nat. Acad. Sci., vol. 35, 1949, pp. 652–655.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

AN APPLICATION OF THE DECOMPOSITION OF A MATRIX INTO PRINCIPAL IDEMPOTENTS

D. W. ROBINSON, Brigham Young University

As a simple application of the decomposition of a (diagonable) matrix into principal idempotent elements, this note provides a proof of the following well-known result: If the nth derivative of a function f exists at α, then it can be computed as the limit of

$$(1) \qquad h^{-n} \sum_{m=0}^{n} \binom{n}{m} (-1)^m f(\alpha + (n-m)h)$$

as h approaches zero.

First, let ϕ be a polynomial such that $\phi^{(m)}(\alpha) = f^{(m)}(\alpha)$ for $m = 0, 1, \cdots, n$.

Since for $n>0$ there exists a function g that is continuous at α with $g(\alpha)=0$ and $f(x)=\phi(x)+(x-\alpha)^n g(x)$, it is clear that the above result will be established providing it is demonstrated for the function ϕ.

Let $A=(\alpha_{ij})$ be a square matrix of order $n+1$, where $\alpha_{ij}=\alpha\delta_{ij}+\delta_{ij-1}$ and δ_{ij} is the Kronecker delta. (A is thus the classical form of a matrix with minimum polynomial $(x-\alpha)^{n+1}$.) If ϕ is expanded about α, then it is clear that $\phi(A)$ has elements

$$\frac{\phi^{(j-i)}(\alpha)}{(j-i)!} \tag{2}$$

for $i\leqq j$ and 0 for $i>j$.

In addition, let $A(h)$ be the diagonable matrix of order $n+1$ defined elementwise for $h\neq 0$ by

$$(\alpha+(i-1)h)\delta_{ij}+\delta_{ij-1}.$$

It is easily verified* that the principal idempotent element $E_k(h)$ of $A(h)$ associated with the eigenvalue $\alpha+(k-1)h$ has elements

$$\frac{(-1)^{j-k}h^{i-j}}{(k-i)!(j-k)!}$$

for $i\leqq k\leqq j$ and 0 otherwise. Therefore, since ϕ is a polynomial, $\phi(A(h))=\sum_{k=1}^{n+1}\phi(\alpha+(k-1)h)E_k(h)$, which has elements

$$\frac{h^{i-j}}{(j-i)!}\sum_{m=0}^{j-i}\binom{j-i}{m}(-1)^m\phi(\alpha+(j-1-m)h) \tag{3}$$

for $i\leqq j$ and 0 for $i>j$. In the limit as $h\to 0$, $A(h)\to A$ and (again since ϕ is a polynomial) $\phi(A(h))\to\phi(A)$. Thus, the limit of (3) is (2), giving in particular the desired result (1) for the function ϕ in the case $i=1$ and $j=n+1$.

* One method is to note that $(zI-A(h))^{-1}=\sum_{k=1}^{n+1}(z-\alpha_k)^{-1}E_k(h)$ has elements $1/\prod_{r=i}'(z-\alpha_r)$ for $i\leqq j$ and 0 for $i>j$, where $z\neq\alpha_r=\alpha+(r-1)h$, and that $E_k(h)=\lim_{z\to\alpha_k}(z-\alpha_k)(zI-A(h))^{-1}$.

A RECURSION FORMULA FOR $\int_0^\infty e^{-t}(\log t)^{n+1}dt$

M. E. LEVENSON, Brooklyn College

Let

$$y=\frac{\Gamma'(x)}{\Gamma(x)}=\frac{d}{dx}\log\Gamma(x).$$

Taking the nth derivative of both sides of $\Gamma'=y\Gamma$, using Leibnitz' rule, we get

$$\Gamma^{(n+1)}=\Gamma^{(n)}y+\binom{n}{1}\Gamma^{(n-1)}y^{(1)}+\cdots+\Gamma y^{(n)}, \tag{1}$$

where

$$\Gamma^{(r)} = \int_0^\infty e^{-t}t^{x-1}(\log t)^r dt$$

and

$$y^{(r)} = \frac{d^{r+1}}{dx^{r+1}} \log \Gamma(x).$$

Setting $x=1$ in (1) and using the formula

$$\frac{d^{r+1}}{dx^{r+1}} \log \Gamma(x) = (-1)^{r+1}r! \sum_{\nu=0}^{\infty} \frac{1}{(x+\nu)^{r+1}} \qquad (r \geq 1),$$

and

$$\Gamma'(1) = \int_0^\infty e^{-t} \log t dt = -\gamma \text{ (Euler's constant)},$$

we get the desired recursion formula

$$\begin{aligned}\int_0^\infty e^{-t}(\log t)^{n+1}dt = &-\gamma \int_0^\infty e^{-t}(\log t)^n dt + \binom{n}{1}\zeta(2)\int_0^\infty e^{-t}(\log t)^{n-1}dt \\ &- 2\binom{n}{2}\zeta(3)\int_0^\infty e^{-t}(\log t)^{n-2}dt + \cdots \\ &+ (-1)^{n+1}n!\zeta(n+1),\end{aligned} \tag{2}$$

where

$$\zeta(s) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^s}.$$

For $n=1$ we get the known result (see this MONTHLY, Jan. 1938, Problem 3766)

$$\int_0^\infty e^{-t}(\log t)^2 dt = \frac{\pi^2}{6} + \gamma^2.$$

For $n=2$ we get

$$\int_0^\infty e^{-t}(\log t)^3 dt = -\gamma^3 - \gamma\frac{\pi^2}{2} - 2\zeta(3),$$

with similar results for $n \geq 3$.

MORE ABOUT LOGARITHMS

C. L. SEEBECK, JR. AND HENRY C. MILLER, JR., University of Alabama

In a recent article on logarithms in the Classroom Notes of this MONTHLY,* a uniqueness proof for the function defined by the given postulates was presented at the graduate-mathematics level. Professor W. L. Duren, Jr. has pointed out the need for such a proof at a lower level to make the presentation of the given logarithm function complete. This article proposes such a proof and points out lines along which the given development could be expanded. The notation, postulates, and preliminary theorems of the original article are assumed to be known.

LEMMA. *If* $\log(a) < \log(b)$, *then* $a < b$.

Proof. Otherwise $a \geqq b$, which would imply $\log(a) \geqq \log(b)$.

Density Assumption. If a and b are two distinct real numbers with $a < b$, there exists a rational number r such that $a < r < b$.

THEOREM. *There is a unique function* log *satisfying the given postulates.*

Proof. If not, then there are two functions log and Log, and a positive real N such that $\log(N) < \mathrm{Log}(N)$, for example. By the density assumption there exists a rational number r such that

$$\log(N) < r < \mathrm{Log}(N).$$

A previously-proved corollary implies that

$$\log(N) < \log(10^r) = \mathrm{Log}(10^r) < \mathrm{Log}(N).$$

And the lemma gives $N < 10^r < N$; an impossible conclusion.

The existence of an inverse function can be established using only the least upper bound postulate. It is interesting to note that even here no continuity concepts are necessary.

THEOREM. *If* y *is any real number, there is a real number* b *such that* $\log(b) = y$.

Proof. Let S be the set of all rational numbers less than y. For all s in S we know that $\log(10^s) = s$. Since there are rational numbers greater than y, the set of all 10^s with s in S is bounded above and hence has a real least upper bound b. Now it is not true that $\log(b) < y$. For, if so, there is a rational number s such that $\log(b) < s < y$, and $b < 10^s$ and b is not an upper bound. Moreover it is not true that $y < \log(b)$. For, if so, there is a rational r not in S such that $y < r < \log(b)$. This implies $10^r < b$ and b is not the least upper bound. Therefore there exists a real b such that $\log(b) = y$.

The following extensions and theorems are suggested. Define the increasing

* C. L. Seebeck, Jr. and John W. Jewett, A development of logarithms using the function concept, this MONTHLY, vol. 64, 1957, pp. 667–668.

logarithm function to the base c by replacing the 10 in postulate 2 by any real $c>1$. Then the following can be proved by the reader.

THEOREM. *If k is any positive real constant, $f(x)=k\log_c(x)$ defines a logarithm function f to some base c', and conversely.*

COROLLARY. $k=\log_{c'}(c)=1/\log_c(c')$.

THEOREM. $f(x)=\int_1^x dt/t$ *is a logarithm function.*

MATHEMATICAL EDUCATION NOTES

Edited by JOHN R. MAYOR, American Association for the Advancement of Science and the University of Maryland, and JOHN A. BROWN, University of Delaware

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE ANNUAL CONFERENCE OF THE NATIONAL COMMISSION ON TEACHER EDUCATION AND PROFESSIONAL STANDARDS

Each year the National Commission on Teacher Education and Professional Standards holds a conference of several hundred persons interested in teacher education. The Commission, commonly called the TEPS Commission, is a unit of the National Education Association. The fifteen members of the Commission are appointed by the Board of Directors of the NEA. This year the TEPS Conference was co-sponsored by the American Association for the Advancement of Science, the American Council of Learned Societies, the Educational Advisory Board of the National Academy of Sciences—National Research Council, and a number of professional education organizations. As a result of sponsorship by the three professional academic societies, some 250 persons from academic departments in colleges and universities participated in the Conference along with about 800 others who were representative of elementary and secondary school teachers, school administrators, professors and deans of education. It seems certain that the 1958 TEPS Conference held at Bowling Green State University was the most widely representative of all levels and fields of instruction of any group of educators ever assembled.

It is not the purpose of the Conferences to approve specific teacher education programs or to pass resolutions. Rather, the discussion each year is centered on a particular theme, and an opportunity is provided for presentation of and debate over issues related to the theme. This year the theme was based on programs of teacher education, both subject matter and professional education. A report of the Conference may be obtained by writing to T. M. Stinnett, National Education Association, 1201 Sixteenth Street, N.W., Washington 6, D. C. A briefer report of the Conference by Loren B. Pope appeared in the New York *Times*' section, Education in Review, for July 6, 1958.

Affiliated with the National TEPS Commission are state commissions in each of the 49 states, which are associated with the state education organizations. The state commissions hold annual state conferences. The themes of the state conferences usually follow the national pattern. While the state TEPS commissions have no official responsibility for teacher certification, these commissions are extremely influential in state-level decisions.

Among the participants in the Bowling Green Conference, from academic departments of colleges and universities, were some 20 mathematicians. Three of these have been invited to make brief statements on their reactions to the Conference for this Department of the MONTHLY. The first two of these statements follow.

REACTIONS TO THE BOWLING GREEN CONFERENCE

P. C. ROSENBLOOM, University of Minnesota

The most striking impression I had, from attending for the first time a meeting of professional educators after long experience in scientific societies, was of the highly developed organizational machinery. The conferees were divided into 32 groups of about 30, each supplied with a chairman, a recorder, and several analysts. There were also roving reporters who moved from group to group, and an administrative staff to reproduce rapidly anything that anyone wished to put in writing. This provided a chance for everyone to take active part in the discussions without being completely swamped by 1,000 other delegates.

There was a tendency for the groups to dilute their statements to vague generalities in order to reach expressions of opinions to which all could subscribe. It would have been better to have each group produce strong majority and minority reports.

It was encouraging to find so many professional educators with whom I could reach an understanding. Some of the speeches, especially those of Margaret Mead and John Goodlad, were excellent.

Probably the most practical result was the establishment of communication between scholars and educators within the state delegations. It is likely that academic people will be invited to participate in policy-making on a state level on matters of teacher education and certification. I hope this will stimulate scholars to take an active interest in effecting the changes they wish, instead of confining themselves to fruitless complaining.

It is also important that mathematicians try to influence educators to take a scientific approach to the solution of the problem of obtaining a maximum output, using quantitative measures, by adapting limited resources in time, manpower, budget, and physical plant. Here is where experts in operations analysis could use their professional competence to improve substantially the administration of the schools.

REMARKS ON THE TEPS CONFERENCE, JUNE 1958

F. A. FICKEN, University of Tennessee

The TEPS Conference was a very welcome first step in narrowing the regrettable schism between educationists and specialists in the various learned subjects. The latter have traditionally declined to accept specific responsibilities in connection with the training in pedagogy of prospective teachers. In our own area we have been concerned primarily with educating more mathematicians and secondarily with giving sound technical training in the uses of mathematics.

One teaching in college may feel that he needs deep knowledge of his subject, enthusiasm for it, an acceptable level of expository ability, and a little experience, while further differences are primarily temperamental and beyond didactic adjustment. At the Conference one began to realize that the situation confronting the secondary and, especially, the primary teacher is rather different from that encountered in the college classroom. Whatever one's basic view on the feasibility of "teaching teachers to teach," he must accept empirically the claim of active teachers that practice teaching and a certain amount of training in "methods" are helpful to them in the classroom. While recognizing the specialized interests of educationists in such matters, mathematicians should be able to give valuable assistance in training teachers in the pedagogy of our subject. One proposal, for example, is that practice teaching in the higher grades, when clear-cut subjects are being taught, should be supervised jointly by educationists and experts in the subject. Much concern is also being expressed that college curriculums in various learned subjects, being intended primarily for prespecialists, are perhaps not too happily adjusted to the needs of those expecting to teach in schools.

Several aspects of the thorny problem of certification were discussed. Many felt that a prospective teacher should have a five-year course of preparation which would include a standard liberal arts program amounting altogether to four-fifths of the work (foreign language to be included, with an academic major and an academic minor), and a program of professional training (including methods, apprentice and practice teaching, and so on) amounting to one-fifth of the work. So demanding a requirement is evidently quite impractical in many states under prevailing economic conditions.

There is no panacea for the problem of inducing more and abler people, with better training, to become teachers. Improved salaries, more propitious working conditions, a more challenging type of preparation, and more sympathic acceptance and support by the community are all clearly necessary for the development of a corps of able and enthusiastic teachers. Taken together, these measures might even be sufficient. Specialists in the learned subjects can help assure themselves a steady supply of able and enthusiastic students by assisting actively in training well-qualified teachers.

MATHEMATICS, SELF-TAUGHT

JOHN D. BAUM, Oberlin College

1. Introduction. The following is the account of an experiment in the teaching of a course in freshman mathematics in the academic year, 1957–1958, at Oberlin College. Very roughly speaking, the students in the course were in class only two-thirds of the school year, while during the other third of the year they studied material in the course on their own. A detailed description of the design of the experiment will follow, but it might be well at this point to give a short history of the motivation for the experiment.

A plan has been under study at Oberlin College for several years which would enable the college to enlarge its student body, utilize its physical facilities, and yet not increase the size of its faculty in the same proportion as the increase in the size of its student body. The plan involves dividing the calendar year into four quarters, and the student body into two equal subcolleges, A and B. During Quarter I, subcollege A would attend classes, subcollege B would be on vacation; during Quarter II, subcollege A would work independently, *i.e.*, away from the campus and with less faculty supervision; subcollege B would attend classes; during Quarter III subcollege A would attend classes, subcollege B would work independently; and during Quarter IV subcollege A would be on vacation, subcollege B would attend classes. Faculty would teach three out of every four quarters, arranged so that three-fourths of the faculty would be on campus at any given time. To date the plan has not been put into operation; however, it is clear that in considering such a plan many questions presented themselves, not the least of which was the following: Can students learn as much if they are out of class one-third of the school year? In an attempt to answer this question insofar as mathematics is concerned, the experiment described here was undertaken.

Funds to support this experiment, as well as similar experiments in biology and psychology at Oberlin College and in a variety of fields at Antioch College and Carleton College were obtained by a grant from the Fund for the Advancement of Education. No student was in two of these experiments at Oberlin simultaneously.

2. Design. The experiment was designed to test the following hypotheses:

a) That students who study freshman mathematics by the usual classroom method will not learn significantly more than students who study on their own for approximately one-third of the school year.

b) That students who study freshman mathematics by the usual classroom method will not gain significantly over students who study independently for approximately one-third of the year in their ability to attack new mathematical concepts.

Four sections of freshman mathematics were devoted to the experiment. (In all there are eight sections of freshman mathematics taught each year.) Students

in all sections are selected randomly. Two of the sections devoted to the experiment were taught in the usual way and were used as control sections. The other two sections were experimental sections. Two instructors were involved in the experiment, Professor Wade Ellis and the author. Each instructor taught one experimental and one control section. There were 78 students involved, 36 in control sections, 42 in experimental sections. Students in all sections attended class from the beginning of school, September 17, 1957, until the beginning of Christmas vacation, December 19, 1957. During this period the experimental sections had a one-week trial period of independent study, the week of November 11–18. From the end of Christmas vacation, January 3, 1958 until the end of Spring vacation, April 8, 1958, the experimental sections studied independently, while the control sections attended class in the usual way. The experimental sections were provided with collateral material—a study guide—which

TABLE I

Date	Experimental Sections	Control Sections	Tests (Date)
Sept. 17, 1957 to Nov. 11, 1957	In Class	In Class	Pretest on First-Semester Final (week of 9/17/57) Learning-Resourcefulness Pretest (week of 9/17/57)
Nov. 11, 1957 to Nov. 18, 1957	Trial Independent-Study Period	In Class	
Nov. 18, 1957 to Dec. 19, 1957	In Class	In Class	Pretest on Second-Semester Final (12/10/57)
Dec. 19, 1957 to Jan. 3, 1958	Christmas Vacation	Christmas Vacation	
Jan. 3, 1958 to Mar. 29, 1958	Independent-Study Period	In Class	First-Semester Final (week of 1/20/58)
Mar. 29, 1958 to Apr. 8, 1958	Spring Vacation	Spring Vacation	"April" Examination (4/8/58)
Apr. 8, 1958 to May 24, 1958	In Class	In Class	Essay Examination by Outside Examiner (5/24/58) Second-Semester Final (week of 5/26/58) Learning-Resourcefulness Test (week of 5/19/58)

contained explanations of difficult material in the text, partially-worked problems from the text, and occasionally additional problems. A schedule of daily assignments was given all sections and it was suggested to the experimental sections that the schedule be adhered to during the independent study period. From the end of Spring vacation until the end of classes, May 24, 1958, all sections attended classes in the usual way. The schedule of class attendance versus independent study as well as the schedule of tests detailed in Section 3, is given in tabular form in Table I.

During the first period, September–December, an attempt was made by both instructors to train the students in the experimental sections to work on their own, and to work together on problems and textual material which proved difficult. It was hoped that during the independent study period, January–April, these students would meet regularly to work together to clarify their individual difficulties. To some extent this hope was fulfilled, though many of the students preferred to work entirely on their own with no help from other students.

The texts used in the course were *Principles of Mathematics* by C. B. Allendoerfer and C. O. Oakley, New York, 1955, Chapters 1–9, and *Calculus with Analytic Geometry* by R. E. Johnson and F. L. Kiokemeister, Boston, 1957, Chapters 2, 4, and 5. The portion of these texts covering during the independent study period was Chapters 6, Section 7 through Chapter 9 of the former text, and Chapter 2 of the latter.

3. Testing. In order to test the hypotheses of the design, various examinations were given. In all cases the same examinations were given to both the experimental and control sections. In all, four measures were used to test the hypotheses; three of these were essentially achievement examinations, the fourth was a measure of ability to work independently on new mathematical concepts. For lack of a better name it was decided to call the latter of these a "learning-resourcefulness" examination. A fifth test was used to determine whether the random selection of students had resulted in control and experimental sections which were truly comparable.

The final examination for the first semester was given at the beginning of the semester and again at the end, and the difference of these two scores for each individual was taken as a measure of the achievement of the individual student during the course. This score, *i.e.* the difference between the initial score and the final score on the first-semester final examination is called simply the first-semester final. A similar technique was followed with the second-semester final. Also, an examination was given at the first class meeting after Spring vacation, when the experimental sections had just finished their independent-study period. This examination covered the material of the independent-study period. This examination is called the April examination in what follows. All three of these examinations were multiple-choice examinations. The following technique was used in scoring these examinations, the purpose being to enable the student to get partial credit on questions: the student was instructed to cross out all

answers he believed wrong (there were invariably four answers to each question, only one of which was correct). The grader then gave three points credit for the correct answer, if it was not crossed out, and minus one for each wrong answer not crossed out. The possible score on each question thus was any integer in the interval $[-3, 3]$. If a student was in doubt as to which of two answers was correct, he could leave them both, and possibly get a partial credit of two points. Students were made aware of the grading system.

The other measure of achievement was an essay-type examination given by an outside examiner and graded by him. No instructions were given the examiner beyond giving him the material covered in the course and asking him to devise a test to measure relative achievement in the subject matter. This examination was given in May, 1958.

The learning-resourcefulness examination was a set of two questions; there were three forms of the examination, with different questions on each form. As with the final-semester examinations, each student was given such a learning-resourcefulness examination at the beginning of the year, September, 1957, and again at the end of the year, May, 1958. He was not given the same form at the beginning as at the end, in order to avoid any practice effect. The difference of the mean scores at the beginning and end for the experimental and control sections was used as a measure of increase in learning resourcefulness. One-third of the experimental section (as well as the control section) was given form 1 of the examination, similarly with forms 2 and 3. The following is a fairly typical question from the learning-resourcefulness examination:

Let S be a collection of elements and let "$\supseteq$" be a relation between pairs of these elements with the following properties:

1. For each a in S, $a \supseteq a$.
2. If $a \supseteq b$ and $b \supseteq c$, then $a \supseteq c$.

Definition: $a \sim b$ if and only if $a \supseteq b$ and $b \supseteq a$.

Definition: $a \supset b$ if and only if $a \supseteq b$ and $a \not\sim b$. (*i.e.* it is false that $a \sim b$).

Prove the following:

I. $a \sim a$.
II. $a \sim b$ implies $b \sim a$.
III. $a \sim b$ and $b \sim c$ imply $a \sim c$.
IV. It is false that $a \supset a$.
V. $a \supset b$ and $b \supset c$ imply $a \supset c$.

The student was given one hour to work on this question; an equal amount of time was given for his other question.

4. Results. The first-semester final was used as a measure of comparability since during the first semester, with the exception of two weeks, both the experimental and control sections had had the same instruction. The difference in the means of the experimental and control sections was small, and according to the t-test occurred 90 per cent of the time by chance alone. Accordingly the

sections were comparable on further tests.

In what follows all differences of scores are expressed as standard score, *i.e.* as a decimal multiple of the standard deviation; all differences are measured for significance by the *t*-test.

On the April examination, the difference of the means was .200 in favor of the control group; such a difference occurs 38 per cent of the time by chance alone and is not considered significant. On the second semester final the difference of the means was .163 in favor of the control group; such a difference occurs 47 per cent of the time by chance alone and is not considered significant. On the essay examination given by the outside examiner the difference of the means was .068 in favor of the control group; such a difference occurs 77 per cent of the tine by chance alone and is not considered significant.

On the learning-resourcefulness examination the difference of the means was .361 in favor of the experimental group; such a difference occurs 13 per cent of the time by chance alone. Usually the 10 per cent level of significance is the largest considered significant; however, since all other scores favor the control group, some attention might well be given this last result.

It thus appears that in an independent-study program in mathematics of of the type described no significant differences in the learning of the material of the course occur. The learning-resourcefulness examination seems to indicate, however, that there may be advantages in the experimental method, namely insofar as the student learns early in his college career to work more successfully on his own.

THE MATHEMATICS PROGRAM AT THE PHILLIPS EXETER ACADEMY

Editors' Note: In the letter submitting the statement to follow, Mr. Ransom Van B. Lynch, The Phillips Exeter Academy, wrote:

"We here at The Phillips Exeter Academy have evolved, in the course of the past 25 years, a four-year secondary-school mathematics program which we believe has great advantages over the traditional curriculum in the United States, and which we are convinced could be used in most college-preparatory mathematics groups. Secondary-school mathematics is certainly not the direct concern of the Association, but it seemed to me that now, when there is so much talk of revising the entire corpus of mathematics instruction, a brief note on Exeter's moderate, conservative, and tremendously effective approach to one segment of this problem might be appropriate."

The editors are in complete accord with Mr. Lynch and are happy to include this statement in the first issue of this Department.

All boys are required to take three years of mathematics. The three-year sequence develops ideas which seem to us a reasonable part of a boy's general

education and the techniques necessary for their understanding. The calculus course in the senior year, elected by three-fourths of the boys, pulls together and consolidates these ideas and techniques so effectively that the student is then ready to go on in the direction dictated by his interests: pure mathematics, the physical sciences, the behavioral sciences. We have found that the calculus course has given to the work of the first three years a focus which has guided us in the elimination of irrelevant detail and of superfluous repetition.

The work of the *ninth grade* is basically a thorough course in elementary algebra. It includes radicals and the general quadratic equation. There is thorough drill in algebraic mechanics and arithmetic with considerable emphasis on the postulational basis of algebra as well as on the recognition of patterns and the manipulation of forms. Considerable variation occurs from section to section in this grade. Boys who start the work with some knowledge of algebra (usually about half a year's work in the eighth grade) are easily able to cover an introduction to demonstrative geometry which goes through the basic congruence theorems. Those who have not studied any formal algebra rarely have time for this unit of geometry or for the unit of numerical trigonometry. We have considered making the geometry a part of the minimum course in grade IX and postponing if necessary the work on irrational numbers.

In the *tenth grade* we carry plane geometry through the conventional work in congruence, rectilinear figures, similarity, the Pythagorean Theorem and areas of rectilinear figures. Solid geometry includes applications of the plane geometry theorems in three dimensions plus some work with the volumes of prisms and pyramids. The study of the circle is postponed until the eleventh grade.

In the tenth grade we also continue the algebra of the ninth grade and expand it to include irrational equations, some work in inequalities, exponents extended to the field of rational numbers, logarithms and the solution of right triangles. Solid geometry appears again when we extend this last item to simple three-dimensional applications. The study of variation—in itself and applied to geometric situations—leads to sets of ordered pairs of real numbers and a thorough study of the linear function, including the analytic geometry of the straight line.

The work in the *eleventh grade* begins with a study of the quadratic function $f(x)=ax^2+bx+c$ and its graph, the parabola. The concept of locus is emphasized in graphing this and other functions. Complex numbers are introduced in order to complete the study of $f(x)=ax^2+bx+c$. We consider the sum and product of the roots, the formation of functions and equations under given conditions maximum and minimum values. We use area formulas and distance-rate-time relations to build quadratic functions. In the study of the circle, introduced at this time, the traditional synthetic approach is supplemented by analytical methods which include the general equation $(x-h)^2+(y-k)^2=r^2$ and simple exercises based on this equation. The study of the intersection of straight lines

and curves leads to the study of tangency, the use of the discriminant, and to work in simultaneous quadratic equations. Locus is discussed and there are some applications to the sphere and the cylinder. In sections where there are both time and inclination a sizeable unit of work is done with the ellipse and the hyperbola. The work in trigonometry in this grade includes the general angle, the laws of sines and cosines, and simple equations and identities based on the reciprocal and Pythagorean relations. There is a unit on the measurement of the circle, some further solid geometry mostly concerned with areas and volumes of cylinders and cones, and brief surveys of arithmetic and geometric progressions. The course ends with a systematic study of the polynomial function with emphasis on the cubic. This requires synthetic division, the factor theorem and finally a method (usually interpolation) for approximating the irrational roots of equations of degree higher than the second, again with emphasis on the cubic.

There are offered in the eleventh grade two special courses for boys entering the Academy in this grade with either two years of algebra and no formal geometry or one year of each. These courses are designed to prepare these boys for the work of the twelfth grade.

In the *twelfth grade* from six to eight weeks are used to complete the analytic trigonometry, including compound and multiple angles. The remainder of the year is given to analytic geometry and calculus.

In the calculus the content, the balance between ideas and techniques, and the degree of rigor conform with the usual practice in the better college-freshman courses, with modifications for two levels of achievement for "slow" and "fast" sections. The formal analytic geometry is introduced as needed; a fair amount is already at the student's disposal from previous courses. Integration of algebraic functions appears early and the definite integral follows at once. Applications of the fundamental theorem are numerous. When possible there is discussion of indeterminate forms, the theorem of the mean and a few "miscellaneous" topics, the choice of which is left to the individual instructor. At the end of this twelfth-year course most boys in the "fast" sections are ready for the second-year calculus course in the colleges of their choice.

Privately printed texts are available through The Exeter Bookstore, Spring Street, Exeter, New Hampshire.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1336. *Proposed by Albert Wilansky, Lehigh University*

Let $f(0)>0$, $f(1)<0$. Prove that $f(x)=0$ for some x under the assumption that there exists a continuous function g such that $f+g$ is nondecreasing.

E 1337. *Proposed by the Student-Faculty Colloquium, Carleton College*

What is the largest value of n, in terms of m, for which the following sentence is true? If from among the first m natural numbers any n are selected, among the remaining $m-n$ at least one will be a divisor of another.

E 1338. *Proposed by V. F. Ivanoff, San Carlos, California*

Let P and P' be a pair of corresponding points of two projectively-related point rows. If the rows are inverted about P and P', respectively, show that the resulting point rows are similar.

E 1339. *Proposed by A. J. Goldman, National Bureau of Standards, Washington D.C.*

Let $f(t)=t-t^3/6+(t^4/24)\sin(1/t)$ for $t>0$. Prove that for $x>0$, $z>0$, $x+z<1$, the relation $f(x+z)\leqq f(x)+f(z)$ holds.

E 1340. *Proposed by Peter Beisswanger, Tübingen, Germany*

Let b_n be the number given by the last $k+2$ digits of a_n, where a_n is the nth term of the Fibonacci sequence $a_0=0$, $a_1=1$, $a_{n+1}=a_{n-1}+a_n$. Show that for each $k\geqq 1$, the sequence $\{b_n\}$ is periodic with $150\cdot 10^k$ as a period.

SOLUTIONS

Selling Flowers

E 1306 [1958, 205]. *Proposed by Jose Gallego-Diaz, Vanderbilt University*

A girl entered a store and bought x flowers for y dollars (x and y integers). When she was about to leave, the clerk said, "If you buy ten more flowers I shall give you all for two dollars, and you will save 80 cents a dozen." Find x and y.

Solution by D. C. B. Marsh, Colorado School of Mines. The flowers apparently cost $12y/x$ dollars per dozen, but the girl was offered $(x+10)/12$ dozen at

an effective rate of $(12y/x-4/5)$ dollars per dozen. (Rate)(Quantity) = Cost gives us

$$(x + 10)(15y - x)/15x = 2.$$

From the common-sense approach, y must be a positive integer less than 2, *ergo* 1, and we can solve for unique positive integral $x=5$. For a more rigorous approach, the above relation may be solved for y to give

$$y = x(x + 40)/15(x + 10),$$

which has the unique positive integral solutions $(x, y) = (5, 1)$ and $(50, 5)$, the latter representing a most unlikely business practice.

Also solved by A. N. Aheart, E. F. Allen, R. V. Andree's engineering class, Raymond Bagley, Leon Bankoff, Edward Barbeau, Merrill Barnebey, J. E. Beam, B. H. Bissinger, A. P. Boblétt, D. A. Breault, J. L. Brown, Jr., W. E. Buker, W. B. Carver, P. L. Chessin, A. G. Clark, Mary Constable, A. G. Davis and John Darraugh (jointly), W. R. Derrick, R. H. Dieter, K. L. Dugal and J. H. Gandhi (jointly), E. S. Eby, E. L. Ellis and D. L. Muench (jointly), David Fink, Rudolf Fitz, J. F. Foley, Eugene Fomolari, Jr., R. J. Freeman, Susan Friedman, W. V. Gamzon, L. O. Girouard, Jr., Michael Goldberg, R. N. Gordon, S. H. Greene, J. W. Haake, J. D. Haggard, Vern Hoggatt, S. P. Holzapfel, R. H. Hou, A. R. Hyde, Elaine Johnson, Udo Karst, Norbert Kaufman, Irving Kay, J. M. Kingston, J. D. E. Konhauser, Sam Kravitz, M. E. Lakser, Peter Landweber, J. A. Larrivee, L. E. Larson, O. E. Lewis, Joe Lipman, Helen Marston, J. H. Means, Mary Moseley, J. B. Muskat, Herbert Nadler, C. S. Ogilvy, Hyman Orlin, Bart Park, W. D. Peeples, Jr., D. J. Peterson, J. P. Phillips, J. L. Pietenpol, C. F. Pinzka, Susan Pyeatt, Arlan Ramsay, A. P. Rhodes, L. A. Ringenberg, Roberta Robinson, Azriel Rosenfeld, J. A. Schumaker, W. B. Stovall, Jr., G. W. Walker, Chih-yi Wang, Harry Weingarten, H. G. Wetzler, and R. H. Wilson, Jr. Late solution by C. H. Cunkle and J. H. Hodges (jointly).

The Mannheim Configuration

E 1307 [1958, 205]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Let P denote the contact of AB with the circle tangent internally to the circumcircle of a triangle ABC and touching the sides AB and AC. If Q is the foot of the perpendicular from P upon AC, show that PQ is equal to the diameter of the incircle of triangle ABC.

Solution by the proposer. Let ρ denote the radius of the circle (ω) touching AB, AC, and the circumcircle (O). Apply the law of cosines to triangle $AO\omega$, in which $\sphericalangle OA\omega = (C-B)/2$, $O\omega = R-\rho$, $OA = R$, and $A\omega = \rho \csc (A/2)$. We obtain $\rho = r \sec^2 (A/2)$. But $AP = \rho \cot (A/2)$. Hence $AP = r \sec (A/2) \csc (A/2)$. Then, since $rs = bc \sin (A/2) \cos (A/2)$, we find that $AP = bc/s$. It follows that $PQ = AP \sin A = (bc \sin A)/s = 2r$.

The proposition is also an obvious consequence of the theorem: If a circle is tangent internally to the circumcircle of a triangle and to two sides of the triangle, the line joining its points of contact with the sides passes through the incenter of the triangle (see N. A. Court, *College Geometry*, 2nd ed., p. 239).

Also solved by Merrill Barnebey, W. B. Carver, J. W. Clawson, Michael Dixon, Jose Gallego-Diaz, Vern Hoggatt, L. M. Kelly and B. M. Stewart (jointly), and D. C. B. Marsh.

An Inequality Connected with a Square

E 1308 [1958, 205]. *Proposed by I. S. Gál, Cornell University*

Let A_1, A_2, A_3, A_4 be the vertices of a square and let P be an arbitrary point in the plane. Show that

$$\sum_{k=1}^{4} A_kP \geqq (1+\sqrt{2}) \max_{(k)} A_kP + \min_{(k)} A_kP$$

and determine the points P for which equality takes place.

Solution by Leon Bankoff, Los Angeles, Calif. Without loss of generality we may assume $A_4P \geqq (A_3P, A_1P) \geqq A_2P$. Then, by Ptolemy's theorem,

$$(A_1P)(A_3A_4) + (A_3P)(A_1A_4) \geqq (A_4P)(A_1A_3),$$

equality holding when A_1, P, A_3, A_4 are concyclic. Since $A_3A_4 = A_1A_4 = A_1A_3/\sqrt{2}$ it follows that $A_1P + A_2P \geqq A_4P\sqrt{2}$. Hence

$$\sum_{k=1}^{4} A_kP \geqq A_4P\sqrt{2} + A_4P + A_2P,$$

equality taking place when P lies on the circumcircle of the square.

Also solved by D. A. Breault, P. L. Chessin, E. L. Ellis and D. L. Muench (jointly), David Fink, Eugene Fomolari, Jr., Jose Gallego-Diaz, Michael Goldberg, J. W. Haake, Vern Hoggatt, A. R. Hyde, C. H. King, J. D. E. Konhauser, Joe Lipman, D. C. B. Marsh, William Moser, Marlow Sholander, Robert Wonderly and David Zeitlin (jointly), and the proposer.

The proposer pointed out that the result can be extended to regular polygons. For the case of a regular hexagon, for instance, the inequality is

$$\sum_{k=1}^{6} A_kP \geqq (2+\sqrt{3}) \max_{(k)} A_kP + \min_{(k)} A_kP,$$

and equality again holds on the circumference of the circumscribed circle. See, I. S. Gál, *A theorem concerning regular polygons*, Buletinul Politehnicii "Gh. Asachi," pp. 98–106 (1948).

Dissection of a Regular Pentagram into a Square

E 1309 [1958, 205]. *Proposed by Martin Gardner, New York, N. Y.*

Dissect a regular pentagram (five-pointed star) into no more than nine pieces which can be reassembled to form a square. Pieces may be turned over.

Solution by H. Lindgren, Patent Office, Canberra, Australia. Eight pieces suffice, and no turning over is required.

The pentagram is cut as in Figure 1, and the pieces rearranged to form two parallelograms as in Figure 2. Each parallelogram is dissected into a rectangle of length s (the side of the square equivalent to the pentagram) by AB equal to s and CD perpendicular to AB. The rectangles, stacked as in Figure 3, form the equivalent square.

FIG. 1

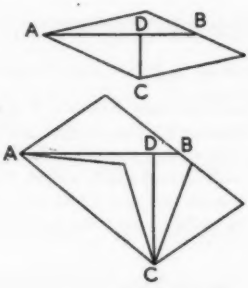


FIG. 2

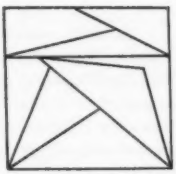


FIG. 3

Also solved by T. Bakos, W. B. Carver, Michael Goldberg, E. R. Herbst, Benjamin Sapolsky, B. M. Stewart, and the proposer.

Bakos and Herbst also obtained solutions involving only eight pieces with none turned over. Carver, Sapolsky, and Stewart obtained solutions involving nine pieces with none turned over.

Evaluation of a Definite Integral

E 1310 [1958, 205]. *Proposed by J. L. Brown, Jr., Pennsylvania State University*

Evaluate $\int_0^{\pi} e \cos x \sin nx \sin(\sin x)dx$, where n is a positive integer.

Solution by B. H. Bissinger, Lebanon Valley College. For $z=\cos x+i \sin x$ we have $\operatorname{Im} e^z=\sum_{k=1}^{\infty}(\sin kx)/k!$. Multiply both sides of this equation by $\sin nx$ and integrate with respect to x from 0 to π. The left member, the proposed integral, is evaluated by the right member, $(1/n!)\int_0^{\pi}\sin^2 nxdx=\pi/(2n!)$.

Also solved by F. M. Carpenter, P. L. Chessin, A. E. Danese, E. L. Ellis and D. L. Muench (jointly), Norman Greenspan, J. W. Haake, D. M. Karns, Norbert Kaufman, J. D. E. Konhauser, David Knee and Gerald Leibowitz (jointly), Joe Lipman, D. C. B. Marsh, D. A. Robinson, Paul Slepian, R. J. Wagner, Chih-yi Wang, J. V. Whittaker, Dale Woods, David Zeitlin, and the proposer.

The integral, in a somewhat generalized form, was located in DeHaan, *Tables of Integrals*, Quallet, *Mathematisches Formelbild*, and Gröbner and Hofrieter, *Integraltafel*. For a similar integral, see E 968 [1951, 338].

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

3951 [1958, 371–2]. *Correction*

The inequality $|2m-n| \leqq 100$ should read $|2n-m| \leqq 100$.

4810. *Proposed by G. B. Dantzig and L. R. Ford, Jr., The Rand Corporation*

Suppose that C_1 and C_2 are convex curves with continuously turning tangents, and that the tangents at the left endpoints are parallel, as are the tangents at the right endpoints. Let C be the locus of points which are the midpoints of line segments joining a point on C_1 with a point on C_2 where the tangents are parallel.

Show that: (a) C is also convex; (b) the midpoint of any segment joining a point on C_1 to a point on C_2 cannot lie below C; and (c) if C_1 and C_2 are placed on parallel planes in 3-space, the lines joining points with parallel tangents generate a convex surface.

4811. *Proposed by Leonard Carlitz, Duke University*

Find the most general solution of the functional equation

$$2^{n/2} f_n(x\sqrt{2}) = \sum_{r=0}^{n} \binom{n}{r} f_r(x) f_{n-r}(x)$$

which is analytic at the origin.

4812. *Proposed by C. Truesdell, Instituto Matematico dell' Universita, Bologna, Italy*

Let m_λ be the smallest positive root of

$$0 = \sum_{q=1}^{\lambda-1} (-1)^{q-1} \sin^2 \frac{q\pi}{\lambda} \cos\left(2\lambda m \sin \frac{q\pi}{2\lambda}\right),$$

$\lambda = 2, 3, \cdots$. Then prove or disprove that m_λ is monotone increasing, $m_2 = \pi\sqrt{2}/8$, $m_3 = \pi(\sqrt{3}-1)/3$, and $m_\lambda \to 1$ as $\lambda \to \infty$.

4813. *Proposed by H. D. Brunk, University of Missouri*

Given a differentiable function $f(x)$ such that to each $x \in (0, 1)$ there cor-

responds a positive integer $k=k(x)$ for which $f^{(n)}(x)=0$ for all $n \geqq k$. Prove f is a polynomial.

4814. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Given that $[n\alpha]+[n\beta]=[n(\alpha+\beta)]$ for all positive integers n. Prove that one of α, β is an integer.

4815. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $a_1, a_2, \cdots, a_n$ be distinct integers. Discuss the reducibility of the polynomial $f(x)=(x-a_1)\cdots(x-a_n)+1$ over the field of rationals.

SOLUTIONS

Angular Spread along a Curve

4759 [1957, 676]. *Proposed by T. K. Pan, University of Oklahoma*

The tangential component of the absolute curvature vector of a unit vector field along a curve on a surface in ordinary space is known as angular spread of the field along the curve and is a generalization of geodesic curvature of a curve. Derive formulas for angular spread generalizing those formulas for geodesic curvature due to Beltrami and Bonnet. (See, e.g., Eisenhart, *Differential Geometry*, 1909, pp. 136, 183.)

Solution by the proposer. Let $S: x^i=x^i(u^1, u^2)$ $(i=1, 2, 3)$ denote a surface of class not less than two in ordinary space. Let $g_{\alpha\beta}$ $(\alpha, \beta=1, 2)$ denote the first fundamental tensor of S and g the determinant $|g_{\alpha\beta}|$. Let $C: u^\alpha=u^\alpha(s)$ denote a curve of class not less than two in S, where s is its arc length. Let v denote an arbitrary but fixed unit vector field of class not less than one with components $v^i=x^i{}_{,\alpha}p^\alpha$. Let ${}_vK_g$ denote the angular spread of v along C for a point P. It is found that

$$ {}_vK_g = \epsilon_{\alpha\beta}p^\alpha p^\beta{}_{,\gamma}\frac{du^\gamma}{ds}, $$

where

$$ \epsilon_{11}=\epsilon_{22}=0, \qquad \epsilon_{12}=-\epsilon_{21}=\sqrt{g}. $$

Suppose the coordinate curves on S form an orthogonal net. Denote by v_δ the unit vector tangent to $u^\beta=\text{const.}$ $(\beta\neq\delta)$. If v_δ is taken as p^α, we obtain a formula for angular spread of v_δ along a curve which makes an angle θ with v_1 as follows:

$$ {}_\delta K_g = -\frac{1}{\sqrt{g_{22}}}\frac{\partial \log\sqrt{g_{11}}}{\partial u^2}\cos\theta+\frac{1}{\sqrt{g_{11}}}\frac{\partial \log\sqrt{g_{22}}}{\partial u^1}\sin\theta. $$

This equation can be cast into the form

$$
{}_{3}K_{\theta} = -\left[\frac{\Delta_2 u^2}{\sqrt{\Delta_1 u^2}} + \Delta_1\left(u^2, \frac{1}{\sqrt{\Delta_1 u^2}}\right)\right]\cos\theta
+\left[\frac{\Delta_2 u^1}{\sqrt{\Delta_1 u^1}} + \Delta_1\left(u^1, \frac{1}{\sqrt{\Delta_1 u^1}}\right)\right]\sin\theta,
$$

where $\Delta_1\phi$ and $\Delta_1(\phi_1, \phi_2)$ denote the differential parameters of the first order and $\Delta_2\phi$ that of the second order.

In any coordinate system, the curves $u^2 = \text{const.}$ and $u^1 = \text{const.}$ are defined respectively by $\phi(u^1, u^2) = \text{const.}$ and $\phi^*(u^1, u^2) = \text{const.}$, which are orthogonal trajectories of each other. Hence we have from the above equation the following generalized Beltrami's formula for angular spread of the unit vector tangent to $\phi(u^1, u^2) = \text{const.}$ along a curve C which makes an angle θ with $\phi(u^1, u^2) = \text{const.}$:

$$
{}_{\phi}K_{\theta} = -\left[\frac{\Delta_2\phi}{\sqrt{\Delta_1\phi}} + \Delta_1\left(\phi, \frac{1}{\sqrt{\Delta_1\phi}}\right)\right]\cos\theta
+\left[\frac{\Delta_2\phi^*}{\sqrt{\Delta_1\phi^*}} + \Delta_1\left(\phi^*, \frac{1}{\sqrt{\Delta_1\phi^*}}\right)\right]\sin\theta.
$$

This formula reduces to Beltrami's formula for geodesic curvature of the curve $\phi = \text{const.}$ when the latter coincides with C.

A generalized Bonnet's formula reads as follows:

$$
{}_{\phi}K_{\theta} = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial u^{\beta}}\left(\frac{\sqrt{g}\, g^{\alpha\beta}\phi_{,\alpha}}{\sqrt{g^{\gamma\delta}\phi_{,\gamma}\phi_{,\delta}}}\right)\cos\theta + \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^{\beta}}\left(\frac{\sqrt{g}\, g^{\alpha\beta}\phi^*_{,\alpha}}{\sqrt{g^{\gamma\delta}\phi^*_{,\gamma}\phi^*_{,\delta}}}\right)\sin\theta.
$$

This formula is an immediate consequence of the generalized Beltrami's formula when we note that

$$
\Delta_2\phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^{\beta}}\left(\sqrt{g}\, g^{\alpha\beta}\phi_{,\alpha}\right)
$$

and that

$$
\frac{\Delta_2\phi}{\sqrt{\Delta_1\phi}} + \Delta_1\left(\phi, \frac{1}{\sqrt{\Delta_1\phi}}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^{\beta}}\left(\frac{\sqrt{g}\, g^{\alpha\beta}\phi_{,\alpha}}{\sqrt{g^{\gamma\delta}\phi_{,\gamma}\phi_{,\delta}}}\right),
$$

where $g^{\alpha\beta}$ denote the normalized cofactors of $g_{\alpha\beta}$ in g.

Orthogonal Polynomials

4766 [1957, 747]. *Proposed by Louis Weisner, University of New Brunswick, Canada*

Prove that the polynomials $g_n(x)$ defined by the generating function

$$
(1 + y^2)^{-1/2} e^{x \arctan y} = \sum_{n=0}^{\infty} \frac{g_n(x)}{n!}\, y^n, \qquad |y| < 1,\ g_0(x) = 1,
$$

are orthogonal over the interval $[-\infty, \infty]$ with weight function $e^{\pi x/2}/(1+e^{\pi x})$. Prove that they satisfy the recurrence relation

$$g_{n+1}(x) = xg_n(x) - n^2 g_{n-1}(x); \qquad n = 0, 1, \cdots; g_{-1}(x) = 0.$$

Solution by A. E. Danese, Mount Morris, N. Y. Let

$$f(x, y) = (1 + y^2)^{-1/2} e^{x \arctan y}, \qquad w(x) = e^{\pi x/2}/(1 + e^{\pi x}).$$

Then

$$I = \int_{-\infty}^{\infty} f(x, y) f(x, z) w(x) dx = (1 + y^2)^{-1/2}(1 + z^2)^{-1/2} \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^{\pi x}} dx,$$

where $a = \arctan y + \arctan z + \pi/2$. Since $0 < a < \pi$, we have, by contour integration

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^{\pi x}} dx = \frac{1}{\sin \pi a}.$$

Then $I = 1/(1 - yz) = \sum_{n=0}^{\infty} y^n z^n$, $|y| < 1$, $|z| < 1$. It now follows, integrating term by term, that

$$\int_{-\infty}^{\infty} g_n(x) g_m(x) w(x) dx = \delta_{mn}(n!)^2.$$

The recurrence relation follows immediately by taking the partial derivatives with respect to y of both sides of

$$f(x, y) = \sum_{n=0}^{\infty} \frac{g_n(x)}{n!} y^n$$

and equating the coefficients of y^n.

Also solved by W. A. Al-Salam and L. Carlitz (two joint solutions), N. Ayirtman, John L. Brown, Jr., T. S. Chihara, E. S. Eby, Peter Henrici, Richard Kelisky, A. K. Rajagopol, Jeff Ritterman, G. Szegö, Chih-yi Wang—and two unsigned.

Editorial Note. More general orthogonal polynomials, of which the above are a special case, can be found in *Higher Transcendental Functions* (New York, 1953) and elsewhere. See, *e.g.*, the Pollaczek polynomials, (vol. 2, pp. 220–221) and the work of Meixner (vol. 3, pp. 273–274).

Generators of a Free Group

4767 [1957, 747]. *Proposed by J. L. Brenner, State College of Washington*

Let a, b, d be integers, w real, $w \geqq 1 + \sqrt{2}$, $d > wa > w^2 b > 0$, $a^2 - bd = 1$. Show that

$$A = \begin{pmatrix} a & b \\ d & a \end{pmatrix}, \qquad B = \begin{pmatrix} a & d \\ b & a \end{pmatrix}$$

generate a free group.

Solution by Joseph Lehner, Michigan State University. This is an immediate corollary of the following theorem of K. Goldberg and M. Newman (*Illinois J. Math.*, vol. I, 1957, pp. 446–448): *Let* $A=(a_{ij})$ *and* $B=(b_{ij})$ *denote rational integral unimodular matrices of order two which are not of finite period. If* a_{12} *is dominant in* A *and* b_{21} *is dominant in* B, *then* A *and* B *generate a free group.* (An element of a matrix is dominant if it is larger in absolute value than any other element of the matrix.)

However, the conclusion holds with the simpler hypothesis, $|a| \geqq 2$, with a, b, d rational integers and $a^2-bd=1$. In view of the theorem quoted above we have only to show that $|b|>|d|$, $|b|>|a|$ or $|d|>|b|$, $|d|>|a|$.

First, $|a| \neq |b|$. For $a=\pm b$ implies $a(a\pm d)=1$ which is impossible since $|a| \neq 1$. Similarly, $|a| \neq |d|$. It is as easy to show that $|b| \neq |d|$. Of the three integers, $|a|$, $|b|$, $|d|$, now shown to be unequal, $|a|$ is not the largest. For, if we had, say $|b|<|d|<|a|$, then $|bd| \leqq (|d|-1)|d| = d^2-|d| \leqq a^2-1-|d| < a^2-1$ (since $d=0$ requires $a^2=1$). But by hypothesis, $|bd| = |a^2-1| = a^2-1$.

It follows that one of $|b|$ or $|d|$ exceeds the other and exceeds $|a|$ and is therefore dominant in both matrices. This completes the proof.

Also solved by Margaret M. LaSalle and the proposer.

A Property of the Regular 26-gon

4768 [1958, 46]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let O be the center of the circumcircle (O) of a regular convex polygon $A_1A_2A_3 \cdots A_{25}A_{26}$ of 26 sides, and let O_1 and O_2 be symmetric to O with respect to the lines $A_{25}A_1$ and A_2A_6. Prove that O_1O_2 is equal to a side of an equilateral triangle inscribed in (O).

Solution by W. J. Blundon, Memorial University of Newfoundland. We may assume the circumcircle has unit radius. Let $\theta=\pi/13$, so that $OO_1=2\cos\theta$, $OO_2=2\cos 2\theta$, and angle $O_1OO_2=4\theta$. By the cosine law,

$$\begin{aligned} O_1O_2^2 &= 4\cos^2\theta + 4\cos^2 2\theta - 8\cos\theta\cos 2\theta\cos 4\theta \\ &= 4 + 2\cos 2\theta + 2\cos 4\theta - 8\cos\theta\cos 2\theta\cos 4\theta. \end{aligned}$$

Upon multiplying by $\sin\theta$ we have

$$\begin{aligned} O_1O_2^2\sin\theta &= 4\sin\theta + (\sin 3\theta - \sin\theta) + (\sin 5\theta - \sin 3\theta) - \sin 8\theta \\ &= 3\sin\theta, \end{aligned}$$

since $\sin 8\theta=\sin 5\theta$. It follows that $O_1O_2=\sqrt{3}$ which is the length of the side of the inscribed equilateral triangle.

Also solved by E. F. Allen, Anders Bager, John B. Baird, M. G. Boyce, D. R. Brillinger, W. B. Carver, J. W. Clawson, Michael Goldberg, F. D. Parker and Ron Graham, J. D. E. Konhauser, A. E. Landry, D. C. B. Marsh, J. B. Muskat, Sister M. Stephanie, Chih-yi Wang, Ralph Wiggins, Roscoe Woods, E. M. Wright, and the proposer.

Analytic Functions with Prescribed Real Part

4769 [1958, 46]. *Proposed by H. Schwerdtfeger, University of Melbourne, Australia*

(1) Find all analytic functions $f(z)=u+iv$ of the complex variable $z=x+iy$, whose real part appears in the form $u=g(x)+h(y)$, where $g(x)$ and $h(y)$ are two real continuous functions of the real variables x and y whose first and second derivatives are continuous.

(2) The same except that $u=g(x)\cdot h(y)$.

Solution by M. F. Neuts, University of Lourain. By the Cauchy-Riemann equations,

$$g'(x) = v_y, \qquad -h'(y) = v_x, \tag{1.1}$$

and the conditions on g and h are sufficient to assure

$$g''(x) + h''(y) = 0. \tag{1.2}$$

From this we find, for $2A$ some real constant,

$$g''(x) = 2A, \qquad h''(y) = -2A, \tag{1.3}$$

$$g(x) = Ax^2 + Bx + C, \qquad h(y) = -Ay^2 + Dy + E, \tag{1.4}$$

where A, B, C, D and E are arbitrary real constants. From (1.1) we determine

$$v(x, y) = 2Axy - Dx + By + F, \qquad F, \text{ a real constant.} \tag{1.5}$$

With $u(x, y)=g(x)+h(y)$ we have finally

$$\begin{aligned} f(z) &= A(x^2 + 2ixy - y^2) + (B - iD)(x + iy) + F + i(C + E) \\ &= Az^2 + \alpha z + \beta, \end{aligned} \tag{1.6}$$

where α and β are arbitrary complex constants. This solves (1).

For $u(x, y)=g(x)\cdot h(y)$ we follow the same line of thought and obtain

$$h(y)\cdot g'(x) = v_y, \qquad -g(x)\cdot h'(y) = v_x, \tag{2.1}$$

$$h(y)\cdot g''(x) + h''(y)\cdot g(x) = 0, \tag{2.2}$$

$$g''(x) - A^2 g(x) = 0, \qquad h''(y) + A^2 h(y) = 0, \tag{2.3}$$

A^2, a real constant. There are three cases according as $A^2>0$, $A^2<0$, or $A^2=0$.

Assume $A^2>0$, thus A can be positive. We have

$$g(x) = Be^{Ax} + Ce^{-Ax}, \qquad h(y) = Ee^{iAy} + Fe^{-iAy}, \tag{2.4}$$

where B and C are real and E and F are complex conjugate numbers. Now the integration of (2.1) is immediate and yields

$$v(x, y) = -i(Be^{Ax} - Ce^{-Ax})(Ee^{iAy} - Fe^{-iAy}). \tag{2.5}$$

With $u(x, y)=g(x)\cdot h(y)$ we obtain $f(z)=u+iv=2BEe^{Ax}+2CFe^{-Ax}$. The conditions on B, C, E, F imply that BE and CF have a real product, so that the

solution can be written:

$$f(z) = \alpha e^{Az} + k\bar{\alpha}e^{-Az}, \tag{2.6}$$

where A and k are real and α is an arbitrary complex constant.

In case $A^2<0$, so that A is pure imaginary, the only change in the above argument occurs at (2.4) where we must now take E and F as arbitrary real numbers with B and C complex conjugate numbers. The resulting solution has the same form as (2.6) but A is pure imaginary.

For $A=0$, $g''(x)=h''(y)=0$ yields

$$g(x) = ax + b, \qquad h(y) = cy + d$$

with a, b, c, d real constants. Following the now familiar argument we obtain easily

$$\begin{aligned} f(z) &= -\tfrac{1}{2}aciz^2 + (ad - bci)z + (bd + ki) \\ &= Diz^2 + (B + Ci)z + (BC/2D + ki), \end{aligned}$$

in which $D(\neq 0)$, B, C, k are real.

Also solved by G. D. Antonio, Robert Breusch, D. R. Brillinger, Emil Grosswald, J. W. Haake, Gerald Leibowitz, E. W. Marchand, D. C. B. Marsh, Imanuel Marx, M. Morduchow, D. E. Myers, D. A. Nelson, M. F. Neuts, M. J. Pascual, B. S. Popov, Nathan Schwid, Chih-yi Wang, and David Zeitlin.

A Divergent Series

4770 [1958, 46]. *Proposed by J. R. Holdsworth and J. R. Smart, China Lake, California*

Determine whether the following series is divergent or not:

$$S = \sum_{n=1}^{\infty} \sin^n n.$$

Solution by John B. Kelly, Michigan State University. We establish divergence by showing that $\limsup |\sin^n n| = 1$.

It is well known that, since $\pi/2$ is irrational, there exist infinitely many rational numbers p/q, with $(p, q)=1$, such that

$$\left|\frac{\pi}{2} - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{1}$$

We may suppose further that q is odd, for the numbers p/q may be taken as the convergents of the simple continued fraction expansion of $\pi/2$ and it is easy to show that two consecutive convergents cannot both have even denominators. From (1) we have $|q\pi/2-p|<1/q$, whence

$$\frac{q\pi}{2} - \frac{1}{q} < p < \frac{q\pi}{2} + \frac{1}{q},$$

so that

$$|\sin p| > \cos\frac{1}{q} > 1 - \frac{1}{2q^2}.$$

Since p/q is near $\pi/2$, we may assume $p/q < 2$ and $p < 2q$. Then

$$(2) \qquad |\sin^p p| > \left(1 - \frac{1}{2q^2}\right)^p > \left(1 - \frac{1}{2q^2}\right)^{2q} \geqq 1 - \frac{1}{q} > 1 - \frac{2}{p}.$$

Thus, by letting n run over the infinite sequence of integers p satisfying (1) with odd q, we see that $\limsup |\sin^n n| = 1$.

Also solved by I. N. Baker, D. R. Brillinger, Paola Comba, Dieter Gaier, Emil Grosswald, Joseph Lehner, Joe Lipman, Andrzej Makowski, Leo Moser, Helmut Salzmann, G. B. Thomas, Jr., E. H. Umberger, L. E. Ward, Sr., and Robert Weinstock. Late solution by Robert Breusch.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Applied Probability. Edited by L. A. MacColl. McGraw-Hill, New York, 1957. v+104 pp. $5.00.

Nine papers on probability and its applications which were presented at the Seventh Symposium in Applied Mathematics sponsored by A.M.S. and O.O.R. in April, 1955. Authors are Paul Lévy, J. L. Doob, William Feller, Eberhard Hopf, Guido Münch, G. K. Batchelor, Mark Kac, S. M. Ulam, and B. O. Koopman. Some of the papers have been (and the others soon will be) reviewed in *Mathematical Reviews*.

SAMUEL GOLDBERG
Oberlin College

The Preparation of Programs for an Electronic Digital Computer. By M. V. Wilkes, D. J. Wheeler, and Stanley Gill. Addison-Wesley, Reading, Mass., 1957, xiv+238 pp. $7.50.

Part I presents as adequate a treatment of programming for the EDSAC, or a similar machine, as is probably possible without the aid of flow charts. The reviewer is at a loss to explain why the authors have chosen to ignore a concept which is as important to programming for digital computers as is "flow charting." In the preface the authors state " . . . the EDSAC is a single-address binary machine, and a recent survey by M. H. Weik shows that, of the *types* of machine(s) currently in use, about 50 per cent use single-address order codes and about 60 per cent work in the binary scale." This statement, though true,

may be somewhat misleading since in actual numbers of machines in use, the single address and/or binary machine is definitely in the minority (Weik, Table II, p. 392). Part II, Specifications of EDSAC Library Subroutines, and Part III, Programs of Selected EDSAC Library Subroutines, are of little general interest unless one wishes to become thoroughly versed in EDSAC coding.

JOHN W. HAMBLEN
Oklahoma State University of Agriculture
and Applied Science and University of Kentucky

An Introduction to the Foundations and Fundamental Concepts of Mathematics. By Howard Eves and Carroll V. Newsom. Rinehart, New York, 1958. xv +363 pp. $6.75.

In the words of the authors: "This book has been written in an attempt to make available to advanced undergraduate students an introductory treatment of the foundations of mathematics and of concepts that are basic to mathematical knowledge." They hope that the exposition may "have value not only to mathematical scholars and scientists but also to philosophers, historians, and others; it is especially to be hoped that potential teachers of mathematics will have an opportunity to study under competent instruction such material as this book contains." Though the book is strongly historical in flavor, the reviewer feels that the historian or philosopher would need some training in mathematics to appreciate this treatment; but certainly the others whom the authors have in mind would be well served.

The first four chapters of this book trace the progress of geometry historically and conceptually from pre-Euclidean through Euclidean and non-Euclidean geometries to the foundations of Hilbert. Here the logical and historical developments go hand in hand as the reader evolves from the primitive to the sophisticated. In the remaining five chapters (Algebraic Structure, Modern Mathematical Method, Real Number System, Sets, Logic and Philosophy) the reason for the order of the topics is not as clear. For instance, in Chapter Seven, the postulational approach to the real number system is first; then comes the development of the number system beginning with the natural numbers and ending with the real and complex numbers. Hamilton's quaternions were previously briefly dealt with in Chapter Five.

The exercises are numerous and very thought-provoking. They are a very important part of the book, for the flavor of mathematics is in them. The theorems and definitions are stated with precision. If a student were to have just one course beyond the calculus, especially if he were preparing to be a secondary-school teacher, a very strong case could be made for using this book, since it combines the perspective of a fascinating historical treatment with a coverage (sometimes somewhat sparse) of most of the fundamental ideas of algebra and geometry.

BURTON W. JONES
University of Colorado

Archimedes. By E. J. Dijksterhuis. Humanities Press, New York, 1957. 422 pp. $12.50.

It is inevitable that this volume be compared with *The Works of Archimedes* by Sir Thomas Heath (1897, recently reissued by Dover, New York). Heath describes his book as "a reproduction of the extant works of perhaps the greatest mathematical genius that the world has ever seen." Dr. Dijksterhuis says that his is "an attempt to bring the work of Archimedes . . . nearer to the understanding and appreciation of the modern world."

The points of difference are important. In the matter of mathematical notation, Dr. Dijksterhuis believes that Heath's use of modern symbols masks the characteristic qualities of the classical argument. Accordingly he has utilized a notation of his own invention which he had found useful in his translation of Euclid. This innovation, however, did not appeal to Dr. van der Waerden who said, in *Science Awakening*, that this is not necessary "provided we take good care *not to use algebraic transformations which cannot immediately be reformulated in the Greek terminology*." In view of the scant acquaintance of American readers with the Greek alphabet, it is unfortunate that Dr. Dijksterhuis used capital letters from this alphabet in his diagrams. Heath did not.

Another important difference in the two books is in the treatment of the material with which Archimedes begins many of his works, theorems discovered by his predecessors and others of his own, essential to the work that follows. Heath compares this to a master piece of strategy. Dijksterhuis believes that it tends to obscure Archimedes' purpose. Accordingly he arranges these theorems in a logically constructed system, coded for reference and preceding his treatment of the text of Archimedes.

As a third point of difference, Dijksterhuis gives a running commentary with each translation. Heath makes sparing use of footnotes, relying on his introductory chapters for fuller explanation.

Which work should a student consult? Both!

VERA SANFORD
State Teachers College,
Oneonta, N. Y.

Théorié Élémentaire des Geométries Non-Euclidiennes. By A. Tresse. Gauthier-Villars, Paris, 1957. 150 pp. 2500 fr.

The subject of this book is inversive geometry. The approach to non-Euclidean geometry is based, essentially, on Poincare's model which is conformal, and therefore reflections correspond to inversions. The treatment in the first part is synthetic, detailed but not very clear. There are many nice figures, two or three on some pages, and not a single specific reference to any of them. The second part is devoted to spherical and hyperbolic trigonometry. The attempt to unify the two seems artificial.

M. S. KNEBELMAN
Washington State College

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Buffalo: The following have been appointed Instructors: Mr. A. R. Bednarek, Mrs. Ruth E. Heintz, Mr. Rudolf Meyer, Mr. S. T. Stern.

Mr. D. S. Adorno, Sylvania Electric Products Co., Waltham, Massachusetts, has been appointed Research Assistant, Department of Statistics, Harvard University.

Associate Professor Frances C. Amemiya, George Pepperdine College, has been appointed Associate Professor at California Western University.

Mr. G. C. Anderson, University of Nebraska, has been appointed Instructor at Hamline University.

Dr. P. M. Anselone, Johns Hopkins University, has joined the staff of the Mathematics Research Center, University of Wisconsin.

Associate Professor H. P. Atkins, University of Rochester, has been appointed Professor at the University of Richmond.

Mr. S. D. Battle, Baylor University, has accepted a position as Engineer with Chance Vought Aircraft, Dallas, Texas.

Assistant Professor W. E. Baxter, Ohio University, has been appointed Assistant Professor at the University of Delaware.

Mr. G. E. Bloch, formerly Ensign in U. S. Navy, has accepted a position with Westinghouse Atomic Power Division, W. Mifflin, Pennsylvania.

Assistant Professor H. F. Davis, University of British Columbia, has been appointed Assistant Professor at the Royal Military College, Kingston, Ontario.

Mr. G. E. Dunne, Bell Telephone Company of Canada, has accepted a position as Construction Engineer, Assaly Construction Co., Ottawa, Ontario.

Associate Professor J. W. Ellis, Florida State University, has been appointed Associate Professor at LSU in New Orleans.

Mr. E. D. Glenney, Piasecki Helicopter Corp., Morton, Pennsylvania, has accepted a position as Senior Tool Engineer in the Pilotless Aircraft Division of the Boeing Airplane Company, Seattle, Washington.

Professor F. S. Harper, Drake University, has been appointed Professor of Actuarial Science at Georgia State College of Business Administration.

Mr. J. F. Hecht, Sr., Design Specialist, Lockheed Missile System Division, Sunnyvale, California, has been promoted to Section Head.

Mrs. Sabrina M. Hecht, Instructor, University of Wichita, has been promoted to Assistant Professor.

Associate Professor F. V. Higgins, Fenn College, has been appointed Assistant Professor of Engineering at Fresno State College.

Mr. H. E. Homesley, Jr., Newport News Shipbuilding and Dry Dock Company, has accepted the position of Training Officer, U. S. Army Chemical Corps School, Fort McClellan, Alabama.

Associate Professor Roberta F. Johnson, Wilson College, has been appointed Associate Professor at Colorado State University.

Associate Professor L. H. Kanter, Drexel Institute, has been appointed Associate Professor at Montclair State College.

Mrs. Arline R. Katz, Ohio State University, has accepted a position as Mathematician with the National Security Agency, Fort Meade, Maryland.

Associate Professor Winfield Keck, Lafayette College, is on sabbatical leave for one year to serve as Research Associate in the Physics Department, Brown University.

Mr. C. A. Keeler, Multnomah Junior College, has been appointed Assistant Professor at Portland State College.

Dr. J. B. Kruskal, University of Wisconsin, has been appointed Assistant Professor at the University of Michigan.

Mr. J. B. Lackey, Ballistics Reduction Group, Ordnance Test Activity, Yuma, Arizona, has been appointed Teaching Fellow at the University of Kentucky.

Mr. J. S. Linnekin, Lebanon Valley College, has accepted a position as Mathematician with the Naval Ordnance Laboratory, Silver Spring, Maryland.

Mr. K. A. MacDonald, University of Vermont, has been appointed Instructor at Idaho State College.

Mr. H. H. Martens, Bell Telephone Laboratories, has been awarded a Communications Development Training Program fellowship at New York University.

Associate Professor A. V. Martin, on leave from the University of New Mexico, is a Visiting Staff Member at the University of Sheffield, England.

Mr. Herbert Nadler, Mt. Vernon Life Insurance Co. of New York, is now with the actuarial department of the Manhattan Life Insurance Company, New York.

Dr. W. E. Nance, Harvard Medical School, is now interning at Vanderbilt University Hospital, Nashville, Tennessee.

Dr. E. T. Parker, University of Michigan, has accepted a position as Mathematician with Remington Rand Univac, St. Paul, Minnesota.

Dr. S. V. Parter, New York University, has been appointed Assistant Professor at Indiana University.

Dr. Barth Pollak, Illinois Institute of Technology, has been appointed Assistant Professor at Syracuse University.

Assistant Professor Pasquale Porcelli, Illinois Institute of Technology, is on a year's leave of absence at the Mathematics Research Center, U. S. Army, Madison, Wisconsin.

Associate Professor L. E. Pursell, Grinnell College, is a Visiting Associate Professor at Purdue University.

Mr. K. E. Ralston, Research Assistant, University of California, Los Angeles, has been promoted to Research Mathematician.

Mr. P. B. Richards, Jack & Heintz, Inc., Cleveland, has accepted a position as Senior Engineering Specialist with Thompson Products, Inc., Cleveland, Ohio.

Mrs. Isabelle P. Rucker, Louisa County School Board, Virginia, has been appointed Supervisor of High School Mathematics in the State, with the Virginia State Department of Education.

Mr. I. D. Ruggles, Iowa State College, has been appointed Assistant Professor at San Jose State College.

Associate Professor W. W. Sawyer, University of Illinois, has been appointed Professor at Wesleyan University, Middletown, Connecticut.

Associate Professor Seymour Schuster, Polytechnic Institute of Brooklyn, has accepted a Visiting Associate Professorship at Carleton College, Northfield, Minnesota, for the academic year 1958–59.

Assistant Professor Berthold Schweizer, San Diego State College, has been appointed Assistant Professor at the University of California, Los Angeles.

Assistant Professor Esther Seiden, Northwestern University, is on leave during the academic year 1958–59 at the Indian Statistical Institute, Calcutta, India.

Assistant Professor G. B. Seligman, Yale University, will be at the Mathematisches Institut der Universität, Munster, West Germany, during the academic year 1958–59, on a Fulbright grant.

Assistant Professor F. A. C. Sevier, Rutgers University, has accepted a position as Senior Mathematician with RCA-Missile and Surface Radars, Moorestown, New Jersey.

Mr. G. J. Simmons, University of Oklahoma, is now with the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor R. C. Simpson, Jr., U. S. Naval Academy, Annapolis, Maryland, has been appointed Associate Professor at St. Lawrence University, Canton, New York.

Associate Professor F. M. Stewart, Brown University, is on leave during the academic year 1958–59 at Imperial College, London, England.

Professor Robert Stoll, Oberlin College, has received an NSF Science Faculty fellowship and will spend the year at California Institute of Technology.

Professor Ethel Sutherland, Longwood College, Farmville, Virginia, has retired after having taught there for 12 years.

Dr. Selmo Tauber, Instructor, University of Kansas, has been promoted to Assistant Professor.

Mr. N. O. Tiffany, Bendix Aviation Corporation, Detroit, Michigan, has accepted the position of Manager, Computer Services, Technical Operations, Inc., Monterey, California.

Dr. B. R. Ullsvik, Illinois State Normal University, has been appointed President of Wisconsin State College, Platteville.

Mr. C. R. Warner, University of Rochester, has been appointed Instructor at the University of Connecticut.

Dr. P. M. Whitman is on a William S. Parsons fellowship from The Johns Hopkins University, Silver Spring, Maryland, and will be at The Johns Hopkins University, Baltimore, during 1958–59.

Dr. J. V. Whittaker, University of California, has been appointed Instructor at the University of British Columbia, Canada.

Professor Emeritus Mary C. Spencer, Newcomb College, Tulane University, died September 7, 1957. She was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-NINTH SUMMER MEETING OF THE ASSOCIATION

The thirty-ninth summer meeting of the Mathematical Association of America was held at Massachusetts Institute of Technology, Cambridge, Massachusetts, from Monday, August 25 through Thursday, August 28, 1958 in conjunction with summer meetings of the American Mathematical Society, the Econometric Society, the Institute of Mathematical Statistics, the Society for Industrial and Applied Mathematics, and the Pi Mu Epsilon Fraternity. There were registered 1232 persons, including 519 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning, and on Wednesday and Thursday afternoons. All sessions were held in Kresge Auditorium. Presiding officers were President G. B. Price, Vice-President G. B. Thomas, and Professors R. M. Thrall, F. B. Hildebrand, and R. D. James. The seventh series of Earle Raymond Hedrick Lectures were delivered by Dr. A. S. Householder of Oak Ridge National Laboratory. The general theme of the meeting was Applied Mathematics, interpreted in a broad sense. The Thursday afternoon meeting was devoted to a description of plans for the further expansion of the activities of the Association which were developed at the Washington Conference on the Future Program of the MAA. The Program Committee for the meeting consisted of M. H. Protter, Chairman; J. H. Curtiss, F. B. Hildebrand, H. W. Kuhn, Morris Newman.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Some Mathematical Problems Arising in Computations with Matrices," Lecture I, by Dr. A. S. Householder, Oak Ridge National Laboratory.

"The Role of the Maximum in Economics," by Professor Robert Solow, Massachusetts Institute of Technology.

"Decentralization of Optimizing Decisions Through a Price System," by Professor T. C. Koopmans, Yale University.

"Markov Chains and Their Applications," by Professor J. L. Snell, Dartmouth College.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Dr. Householder.

"Plasticity and the Mechanics of Structures," by Professor H. J. Greenberg, Institute of Mathematical Sciences, New York University.

"Satellite Orbits," by Professor S. P. Diliberto, University of California, Berkeley.

"On Mean Value Theorems in the Theory of Elasticity," by Professor J. B. Diaz, University of Maryland.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Dr. Householder.

Business Meeting of the Association.

"The Clock Paradox in Relativity Theory," by Professor Alfred Schild, University of Texas and Hughes Research Laboratory.

"Mathematical Problems of Boundary Layer Theory," by Professor K. O. Friedrichs, Institute of Mathematical Sciences, New York University.

"Network Analysis of Thermal Conduction Problems," by Professor Richard Duffin, Duke University and Office of Ordnance Research.

FOURTH SESSION OF THE ASSOCIATION

"The Role of Numerical Analysis in an Undergraduate Program," by Professor G. E. Forsythe, Stanford University.

"The Introduction of Numerical Analysis," by Professor John Todd, California Institute of Technology.

"The Leading Ideas of Information Theory," by Professor C. E. Shannon, Massachusetts Institute of Technology.

FIFTH SESSION OF THE ASSOCIATION

"Report on the Washington Conference of the Mathematical Association of America." Panel: G. B. Price, Chairman; E. A. Cameron, H. M. Gehman, R. D. James, J. R. Mayor, Rothwell Stephens.

SPECIAL SESSION ON FILMS

On Wednesday evening, there was shown a colored motion picture film in which Professor George Springer of the University of Kansas taught a lesson in arithmetic. This film has been prepared for the use of the United States Armed Forces Institute. A spirited discussion followed the showing of the film in which its virtues and its defects were pointed out.

Two films prepared by the Physical Science Study Committee were also shown.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Hayden

Library Lounge with seventeen members present. Among the more important items of business transacted were the following:

It was voted that future annual meetings are to be scheduled during the last week in January and are to be primarily hotel meetings. The meeting of January 1960 is to be held in Chicago at the Hotel Conrad Hilton and the meeting of August 1960 is to be held at Michigan State University, East Lansing.

The Board instructed the Secretary-Treasurer to prepare amendments to the by-laws of the Association, (1) providing that the office of Secretary-Treasurer be separated into the two offices of Secretary and Treasurer to be held by separate persons, (2) clarifying the status of the Vice-Presidents, and (3) clarifying the method of amending the by-laws. If these amendments are adopted, Professor H. M. Gehman is to continue as Treasurer, and the Board is to elect a Secretary of the Association.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President G. B. Price presiding. The Secretary-Treasurer reported that the membership of the Association was 7770 on August 20, an increase of over 10 per cent since the corresponding date last year.

Dr. J. R. Mayor reported for the Committee on Secondary School Lecturers, and Professor G. B. Price reported for the Committee to Study the Activities of the Association.

At the conclusion of the Thursday afternoon session, the following motion presented by Professor L. W. Cohen was adopted: "It is the sense of this meeting that the resolutions adopted by the Washington Conference outline a proper and a promising program for the future activities of the MAA and the meeting lends its full support to the officers of the Association in vigorously carrying this program to completion."

MEETING OF SECTION OFFICERS

A meeting of representatives of the Sections of the Association was held on Tuesday evening in the Hayden Library Lounge. Fifty persons were present representing twenty-four Sections.

The role of the Sections in the MAA was discussed, particularly what the Sections expect of the Association, and what the Association expects of the Sections. Reports were given of special projects in which some Sections are now involved.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. Invited addresses were given by Professors B. J. Pettis, Eldon Dyer, Walter Rudin, O. M. Nikodym, and Jürgen Moser.

The Econometric Society met from Tuesday to Thursday, the Institute of Mathematical Statistics from Monday to Thursday, and the Society for Industrial and Applied Mathematics from Monday through Wednesday. The Pi Mu Epsilon Fraternity held a breakfast meeting on Wednesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: G. B. Thomas, Jr., Chairman; J. H. Curtiss, Philip Franklin, H. M. Gehman, F. B. Hildebrand, Norman Levinson, Hartley Rogers, Jr., R. D. Schafer.

Registration headquarters was located in the lobby of Kresge Auditorium. Dormitory and cafeteria accommodations were provided by MIT. Due to the large registration, dormitory accommodations were also provided by Boston University. The employment register and the textbook exhibits were on display near the registration area.

Arrangements were made for tours of the Boston area, of the MIT Computation Center, of the MIT Nuclear Reactor, and of the plant of the Addison-Wesley Publishing Company. The athletic and recreational facilities of MIT were available without charge. The annual SIAM beer party was held on Tuesday evening. A tea and reception given by the administration of MIT was held on Wednesday afternoon at the President's House.

A resolution of thanks prepared by Professor M. R. Hestenes, and adopted by the participating organizations, expressed to the Massachusetts Institute of Technology and to its Department of Mathematics appreciation for the great privilege of holding the summer meetings at the Institute and for their graciousness as hosts.

HARRY M. GEHMAN, *Secretary-Treasurer*

PROPOSED AMENDMENTS TO THE BY-LAWS OF THE M.A.A.

At the annual business meeting of the Association to be held at the University of Pennsylvania, Philadelphia, Pennsylvania on Friday, January 23, 1959, motions will be made to amend the By-Laws as follows:

1. That Article III, Section 1, be amended by replacing "a Secretary-Treasurer" by "a Secretary, a Treasurer."

2. That Article III, Section 3, be amended by replacing "the Secretary-Treasurer" by "the Secretary and the Treasurer."

3. That Article III, Section 7, be amended by replacing the second sentence by: "This committee shall consist of four members, including the Secretary and the Treasurer."

4. That Article III, Section 8 (d), be amended by replacing "a Secretary-Treasurer" by "a Secretary, a Treasurer" and by replacing "(other than the Secretary-Treasurer)" by "(other than the Secretary and the Treasurer)."

5. That Article III, Section 10 be amended by replacing the first sentence by: "In the absence of the President, the First Vice-President (or in his absence the Second Vice-President) shall have and exercise the powers of the President."

6. That Article III, Section 11 be amended to read as follows: "The Secretary shall have the usual duties pertaining to his office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and of special meetings, and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President."

7. That a new section, to be numbered Article III, Section 12, be inserted in the By-Laws to read as follows: "The Treasurer shall have the usual duties pertaining to his office, including the collection of dues and the supervision and safekeeping of the funds of the Association."

8. That Article IV, Section 3 be amended by replacing "the Secretary-Treasurer" by "the Secretary."

9. That Article VIII, Section 1 be amended by adding this sentence: "The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association."

10. That Article VIII, Section 2 be amended by replacing "the Secretary-Treasurer" by "the Secretary."

HARRY M. GEHMAN, *Secretary-Treasurer*

ITINERARIES OF VISITING LECTURERS, 1958–59

The itineraries of two of the visiting lecturers are listed below. The itineraries of the others will be published in the MONTHLY as soon as they are available.

Marguerite Lehr

Dartmouth College	Hanover, N. H.	Nov. 3–4
Smith College	Northampton, Mass.	Nov. 5
Mount Holyoke College	So. Hadley, Mass.	Nov. 6
Univ. of Massachusetts	Amherst, Mass.	Nov. 7–8
Vassar College	Poughkeepsie, N. Y.	Nov. 10–11
St. Lawrence Univ.	Canton, N. Y.	Nov. 12–13
Clarkson College of Technology	Potsdam, N. Y.	Nov. 14–15
Denison Univ.	Granville, Ohio	Mar. 16–17
Kenyon College	Gambier, Ohio	Mar. 18–19
Eastern Michigan College	Ypsilanti, Mich.	Mar. 20–21
Wayne State Univ.	Detroit, Mich.	Mar. 23
Iowa State Coll.	Ames, Iowa	Mar. 25–26
Gustavus Adolphus College	St. Peter, Minn.	Mar. 31
Carleton College	Northfield, Minn.	Apr. 1–2
St. Olaf College	Northfield, Minn.	Apr. 3
Mankato State College	Mankato, Minn.	Apr. 6–7
St. Mary's College and College of St. Teresa	Winona, Minn.	Apr. 8–10

L. H. Loomis

McGill Univ., Univ. of Montreal, Univ. of Ottawa, Carleton Univ.	Montreal, Canada	Apr. 1–4
Douglas College	New Brunswick, N. J.	Apr. 6–7
Connecticut College	New London, Conn.	Apr. 8
Wesleyan Univ.	Middletown, Conn.	Apr. 9–10
Univ. of Rochester	Rochester, N. Y.	Apr. 13–14
Oberlin College	Oberlin, Ohio	Apr. 15–16
Western Michigan Univ.	Kalamazoo, Mich.	Apr. 17
Univ. of Wisconsin	Madison, Wis.	Apr. 20
Coll. of St. Thomas, Macalester Coll., Coll. of St. Catherine, Hamline Univ., Univ. of Minnesota	Minneapolis, Minn.	Apr. 21–25
Indiana Univ.	Bloomington, Ind.	Apr. 27–28
Miami Univ.	Oxford, Ohio	Apr. 29–30

SLAUGHT MEMORIAL PAPERS

The Committee on Slaught Memorial Papers now invites contributions to this series of expository pamphlets which are published as supplements to the MONTHLY. Authors should keep in mind the general Association membership for which the series is intended. The normal length is 60–120 printed pages, or 90–180 double-spaced typewritten pages. Authors of partially-completed manuscripts are invited to submit them for preliminary consideration by the Committee, which presently consists of Professors R. C. Buck, R. D. James (ex officio), R. D. Schafer (Chairman), and F. B. Wright. Papers may be submitted to any member of the Committee or to the editor of the MONTHLY.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 113 persons have been elected to membership by the Board of Governors on applications duly certified.

HARRY P. ALLEN, Student, Brooklyn College.

RASHID A. R. ALSALIHI, B.S. (Iraq) Instr. and Registrar, College of Agriculture, Baghdad, Iraq.

SYED ASADULLA, M.S. (Karachi) Chairman, Faculty of Science, Syed Raza Ali College, Karachi, Pakistan.

ISABEL C. AVERILL, A.B. (Mt. Holyoke) Teacher and Coordinator, Irondequoit High School, Rochester, New York.

DOV AVISHALOM, M.S. (Jerusalem) Lecturer, Bar-Ilan University, Ramath-Gan, Israel.

DONALD J. BAKER, JR., B.S. (Stanford) Grad. Student, Cornell University.

RICHARD J. BAKER, A.B. (Holy Cross) Ensign, U.S.N., San Francisco, California.

ALICE G. BARNHOLDT, B.S. (St. Cloud S.C.) Instr., Rochester Senior High School, Minnesota.

JAMES R. BAUGH, B.A. (U.C.L.A.) Grad. Student, University of California, Los Angeles; Mathematician, Douglas Aircraft Co., Long Beach, California.

RUBY BAXTER, M.A. (Illinois) Teacher and Registrar, Shimer College.

KEITH H. BELCHER, Student, Interamerican University.

EDWARD C. BITTNER, B.S. (Fenn) Mathematician, N.A.C.A., Cleveland, Ohio.

PETER BOITA, M.A.Ed. (Amer. Inter.) Chairman of Dept., Ludlow High School, Massachusetts.

C. MILTON BROSTROM, M.A. (Minnesota) Asst. Professor, Gustavus Adolphus College.

ROBERT E. CARON, Student, Eastside High School, Paterson, New Jersey.

VIRGINIA R. CARTER, A.B. (Lincoln Memorial) Teacher, Dobyns-Bennett High School, Kingsport, Tennessee.

THOMAS E. CAYWOOD, Ph.D. (Harvard) Partner, Caywood-Schiller Associates, Chicago, Illinois.

OLGA CHRISTOFIDES, B.S. (Athens) Instr., Pancyprian Gymnasium, Nicosia, Cyprus.

DELIGHT T. CLAPP, M.A. (Catholic Univ.) Chairman of Dept., Wheaton High School, Maryland.

LESLIE L. COCHRAN, M.S. (Iowa S.C.) Mathematician, Sperry-Rand Corp., St. Paul, Minnesota.

DENNIS COOKE, M.A. (Oxford) Visiting Professor, North Carolina State College.

ROBERT CORTELL, Ph.D. (New York) Asso. Professor, City College of New York.

THOMAS COURTNEY, Student, Texas Technological College.

WILLIAM CRAIG, Ph.D. (Harvard) Asso. Professor, Pennsylvania State University.

MRS. ALICE N. DANIEL, M.S. (Iowa) Instr., Drake University.

KENNETH S. DAVIS, M.S. (Southern California) Teaching Asst., Stanford University.

RAYMOND A. DIBRELL, JR., M.A. (Texas) Member of Technical Staff, Hughes Aircraft Co., Fullerton, California.

FRANK H. DITTO, B.S. (Roosevelt) Grad. Student, Northeastern University.

HERBERT C. DUNKLEY, Actuarial Vice President, North American Life and Casualty Co., Minneapolis, Minnesota.

JOHN A. DUTTON, B.S. (Wisconsin) Res. Asst., University of Wisconsin.

R. BRUCE EMMONS, A.B. (Nebraska) Physicist, Philco Corp., Philadelphia, Pennsylvania.

ANN FAREK, M.A. (Texas) Head of Dept., Laredo Junior College.

NORMAN S. FERTEL, Student, Tulane University.

DAVID FINK, Student, Newark College of Engineering.

VERNON J. FOWLER, M.S. in E.E. (Illinois) Advanced Res. Engr., Sylvania Electric Products Co., Bayside, New York.

JAMES D. FRITTS, Teacher, Gordon Technical High School, Chicago, Illinois; Student, DePaul University.

RICHARD A. GAMMILL, JR., B.S. (Centenary) Jr. Engr., Philco Corp., Philadelphia, Pennsylvania.

C. E. JOHN GERRIETS, M.A. (Stanford) Asst. Professor, Utah State University.

EARL H. GILMORE, Ph.D. (California) Asso. Professor, Oklahoma State University.

EARL GRANT, Nuclear Spec., S/Sgt., U.S.A.F.; Student, University of California Extension, Berkeley.

DANIEL T. GREEN, M.S. (Catholic) Mathematician, Naval Proving Ground, Virginia.

PETER C. GREINER, Student, University of British Columbia.

ARTUR GRIGORI, Ph.D. (Tubingen) Instr., St. Bonaventure University.

SIDNEY L. GULICK, III, Student, Oberlin College.

CHARLES D. GUSTAFSON, B.S. (Abilene) Grad. Student, Abilene Christian College.

CLELA D. HAMMOND, M.A. (California, Berkeley) Instr., El Camino College; Lecturer, University of California, Los Angeles.

MARTIN HARROW, B.S. (McGill) Lecturer, Sir George Williams College.

REV. THOMAS HARTMAN, M.S. (S.U. of Iowa) Professor, St. Benedict's College, Kansas.

RICHARD G. HETHERINGTON, M.S. (Wisconsin) Fellow, University of Wisconsin.

PAUL R. HILDEBRANDT, M.A. (Columbia) Asso. Mathematician, System Development Corp., Santa Monica, California.

LOWELL A. HINRICHS, B.A. (Oregon) Mathematician, Naval Ordnance Lab., Corona, California.

WILLIAM E. HOLMSTROM, Student, Gannon College.

HARRIS R. HOLT, A.B. (Kenyon Coll.) Asso. Mathematician, Johns Hopkins Applied Physics Laboratory.

THOMAS C. HOLYOKE, Ph.D. (Ohio State) Fellow, University of California, Berkeley.

DOUGLAS F. JOHNSTONE, Supervisor, Actuarial Dept., Standard Life Assurance Co., Montreal, Canada.

ROBERT E. KALABA, Ph.D. (New York) Mathematician, Rand Corp., Santa Monica, California.

ABRAHAM KAREN, M.S. (New York) Sr. Physicist, Reeves Instrument Corp., New York, New York.

BELLE KEARNEY, M.A. (Columbia) Asst. Professor, Montclair State College.

SAMUEL M. KEATHLEY, M.A. (Alabama) Loads Engr., Chance Vought Aircraft, Dallas, Texas.

WILLIAM J. KENNEDY, JR., Student, Whitman College.

RICHARD M. KERSTEIN, B.M.E. (Poly. Inst. of Brooklyn) Engineer, Sperry Gyroscope Co., Great Neck, New York.

JUDITH F. KNEEN, A.B. (Bryn Mawr) Grad. Student, Bryn Mawr College.

WILLIAM J. LARKIN, III, M.Ed. (Xavier) Instr., Xavier University.

DONALD T. LEE, Student, University of British Columbia.

LIONEL W. LEVIN, B.S. (Poly. Inst. of Brooklyn) Programmer, Gulf Research & Development Co., Pittsburgh, Pennsylvania.

JULIA B. LINN, M.S. (Chicago) Instr., University of Illinois.

MICHAEL F. LIPP, B.S. (Poly. Inst. of Brooklyn) Teaching Asst., Stanford University.

PATRICK W. MACKAY, B.A. (Texas) Private, U.S.A., Milford, Connecticut.

PHILIP MACLASKY, M.A. (Pennsylvania) Teacher, Central High School, Philadelphia, Pennsylvania.

STANLEY P. MANN, Student, University of Oklahoma.

FRANK C. MAY, Student, University of British Columbia.

HARALD S. MCGLADDERY, Student, University of British Columbia.

JOHN E. MILLER, M.S. (M.I.T.) Engineer, Systems Development Corp., Dayton, Ohio.

JAMES MISHO, M.A. (Harvard) Mathematician, Harvey University.

FLETCHER A. MOSELEY, M.A. (Texas Tech) Instr., West Texas State College.

JOSEPH M. MOSER, M.A. (St. Louis) Grad. Fellow, St. Louis University.

KATHARINE E. O'BRIEN, Ph.D. (Brown) Head of Dept., Deering High School, Portland, Maine.

CLARENCE E. OLANDER, M.A. (Minnesota) Asso. Editor of Math., Scott, Foresman & Co., Chicago, Illinois.

PATTI N. OSBORN, Student, State College of Washington.

ANTONY S. V. PARKER-JERVIS, Teacher, St. Georges School, Vancouver, Canada.

ROBERT PERKUS, M.S.E. (C.C.N.Y.) Teacher, Brooklyn Technical High School, New York.

HAROLD E. PETERSEN, M.A. (Kansas City)

Chairman of Dept., Rockhurst High School, Kansas City, Missouri.

CHARLOTTE A. POTTER, M.A. (Columbia) Teacher, The Brearley School, New York.

CLIFFORD R. QUALLS, Student, University of California, Riverside.

ALICE B. RABON, M.Ed. (South Carolina) Head of Dept., Brookland-Cayce High School, South Carolina.

BERNARD RASOF, Ph.D. (Calif. Inst. of Tech.) Professor, State University College on Long Island.

NATHANIEL R. RIESENBERG, Student, Lafayette High School, Brooklyn, New York.

WILLIAM A. RILEY, JR., Student, University of Kentucky.

J. VINCENT ROBISON, M.S. (Oklahoma S.U.) Asst. Professor, Oklahoma State University.

MRS. MARY G. RODRIGUEZ, M.A. (Virginia) Instr., Southwestern College, Tennessee.

WILLIAM H. RUPLEY, Student, Stanford University.

J. NORMAN C. SHARP, M.A. (Toronto) Head of Dept., Upper Canada College.

DAVID SHELUPSKY, Student, The College of the City of New York.

NORMAN A. SHENK, II, Student, The Rice Institute.

FRANCIS J. SHOLOMSKAS, B.S. (Temple) Instr., Temple University.

ROBERT L. SILVERMAN, M.A. (Michigan) Mathematician, Office of Quartermaster General, Washington, D. C.

SISTER ROSE ALICIA, C.S.J., M.S. (Coll. of St. Rose) Teacher, St. Mary's Institute, New York.

ALFRED L. SLATER, M.S. (Calif. Inst. of Tech.) Instr., Los Angeles City College.

EUGENE P. SMITH, M.A. (Ohio S.U.) Chairman of Math. Area, University School, Ohio State University.

REV. JAMES F. SMITH, S.J., M.S. (Catholic) Grad. Student, Catholic University.

GERALD STRAHS, Student, Cooper Union.

MALCOLM E. TURNER, M.S. (N. Car. S.C.) Asso. Professor, Medical College of Virginia.

LOWELL T. VAN TASSEL, B.S. (Minnesota) Instr., San Diego City Schools.

THEODORE H. VOTH, M.S. (Tulsa) Asso. Professor, Tougaloo Southern Christian College.

KENNETH T. WALLENIUS, M.A. (Southern California) Instr., U. S. Naval Postgraduate School.

SANDRA WEINSTEIN, M.S. (Wisconsin) Grad. Student, University of Wisconsin.

JOSEPH S. WHOLEY, M.A. (Harvard) Instr., Newton College of the Sacred Heart.

EDWARD R. WILLIAMS, B.A. (Bowdoin) Jr. Scientist, AVCO Manufacturing Corp., Rowley, Massachusetts.

HOWARD WILSON, B.E.E. (Geo. Washington) Asst. Instr., University of Maryland.

WILLIAM WOOTON, M.A. (Occidental) Instr., Pierce Junior College.

EDWARD W. WROBBEL, Technician, Gates Engineering Co., Wilmington, Delaware.

MARGARET ZIPP, M.A. (Pittsburgh) Instr., Essex Community College.

WILLIAM L. ZLOT, Ph.D. (Columbia) Instr., City College of New York.

THE APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Baylor University, Waco, Texas, on April 18–19, 1958. Professors Martin Wright and E. K. McLachlan presided at all sessions. There were 125 persons registered for the meeting, including 90 members of the Association.

Officers chosen for 1958–1959 were E. K. McLachlan, Baylor University, Chairman; W. T. Guy, University of Texas, Vice Chairman; C. R. Sherer, Texas Christian University, Secretary-Treasurer. The Section decided to participate in the National High School Mathematics Contest next year on a trial basis. Dr. H. J. Ettlinger, The University of Texas, was made chairman of the committee in charge.

By invitation of the Section, Professor Louis J. Brand, The University of Houston, delivered an address entitled "Some Aspects of Dimensional Analysis." The abstract of this address follows:

In volume 1, number 1 (1957), of the *Archive for Rational Mechanics and Analysis*, Louis Brand proved the pi theorem of dimensional analysis in the form which gives explicit values for the dimensionless products. This form of the pi theorem is used to solve Lord Raleigh's problem of the hot ball in a stream of liquid, first with four, then with three fundamental units, and affords a simple explanation of the paradox that added physical knowledge give a less explicit result. Various aspects of dimensional analysis are then considered: dimensional checks, integral formulas, and the solution of certain ordinary and partial differential equations.

The following papers were presented:

1. *On the method of undetermined coefficients*, by Dr. Arlen Brown, Rice Institute.

The familiar operator-theoretic approach to a linear differential equation with constant coefficients is examined from a slightly different point of view. Attention is focused on the so-called method of undetermined coefficients and the efficacy of that method is shown to be equivalent to a familiar theorem in linear algebra.

2. *A Rodriguez formula*, by Professor F. J. Palas, Southern Methodist University.

The sequence of polynomials $\{T_{kn}(x)\}$ generated by $g(x, t)=(1-t)^{-1}\exp[x^k(1-(1-t)^{-1})]$ is shown to satisfy a recurrence relation. The polynomial $T_{kn}(x)$ can be represented by a Rodriguez formula which is used to show that $T_{kn}(x)$ has a moment property closely related to orthogonality. The formula also permits $T_{k(n+1)}(x)$ to be represented in terms of $T_{kn}(x)$. In addition, the fact that $T_{kn}(x)$ satisfies a differential equation of order $k+1$ is established by means of the recurrence relations.

3. *The inversion of a certain matrix using generating functions*, by Professor R. P. Kelisky, University of Texas.

If $\{x_n\}$ and $\{z_n\}$ are sequences such that, symbolically, $(x+1)^n-x_n=z_{n-1}$ for $n\geqq 1$, then it is possible to express x_n as an explicit function of z_k, $0\leqq k\leqq n$. The inversion is accomplished by assuming the existence of a certain kind of generating function for the given sequences. One finds that $(n+1)x_n=(1/z)[B_{n+1}(z)-b_{n+1}]$; the powers of z on the left side are to be interpreted symbolically, b_n is the nth Bernoulli number, and $B_n(z)$ is the nth Bernoulli polynomial.

4. *Congruences and a certain geometrical construction*, by Professor R. E. Greenwood, University of Texas.

D. J. Behrens in the *Mathematical Gazette*, vol. 41, May 1957, page 101, proposed the following problem:

Given $(2n+1)$ points evenly spaced on the circumference of a circle, n chords are drawn connecting $2n$ points by pairs, and a tangent line is drawn at the $(2n+1)$st point. Is it possible to get a construction so that no two of the chords are of the same length and no chord is parallel to any other chord of the tangent?

The problem is reformulated in number theoretic and congruence terms and an algorithm is developed which will yield a construction if $(2n+1)$ is a prime and n is odd, $n>2$.

5. *Properties of functions which uniformize certain classes of Riemann surfaces*, by Mr. H. B. Curtis, Jr., Rice Institute.

Three classes of open simply connected Riemann surfaces are considered. Forms and properties of the uniformizing function and its derivative are obtained for surfaces of each class.

6. *A theorem on determinants*, by Dr. W. E. Roth, Austin, Texas.

The following proposition has appeared in several places in literature and still is not well known:

If $A=(A_{ij})$ is an $r\times r$ matrix whose elements A_{ij}, $i, j=1, \cdots, r$, are n by n matrices with elements in the field F and are commutative in pairs then the determinant

$$|A|_{rn} = |\sum(-1)^{e_i} A_{1i_1} \cdots A_{ri_n}|_n$$

where e_i is the number of inversions in the sequence $i_1, \cdots, i_r$, and the summation runs over the r-factorial permutations of the integers $1, \cdots, r$. That is, the determinant of order rn is expressible as one of order n.

7. *The behavior of functions harmonic and positive in an angular neighborhood*, by Mr. Daniel Weiser, Rice Institute.

The growth, as w approaches w_0, of a function which is positive and harmonic in a simply connected region D and zero on the boundary except at w_0, a singularity of the function, is determined by the behavior of the function $w=f(z)$ which maps the upper half plane conformally onto D. Inequalities are obtained which characterize $w=f(z)$.

8. *A mechanical model which approximates the sum of an annuity*, by Professor R. C. Osborn, University of Texas.

When a ribbon is wound on a spool, the length of arc of the spiral so formed approximates the sum of an annuity. This paper compares the length of arc to the sum of an annuity for various rates and times, describes the relation of the length of the arc to the sum of the annuity, and describes the relation between the interest rate and two other factors, namely, diameter of spool and thickness of ribbon.

9. *Concerning some types of connected point sets*, by Mr. J. R. Boyd, Chance Vought Aircraft, Dallas, Texas.

In Volume 47 of the Bulletin, P. M. Swingle defined the term *irreducible connected closure* between two points. This paper established the relationship between this type of connected set and connected sets that are indecomposable, widely connected, and irreducibly connected between two points. Also, Swingle's definition is generalized to irreducible connected closure about a point set and necessary and sufficient conditions are established for such a set to be irreducibly connected about a point set.

10. *Integrable functions*, by Mr. J. L. Cornette, University of Texas.

The function f is integrable over the interval (a, b) if and only if there exists a sequence f_n of step functions on (a, b) such that 1) $\lim f_n=f$ almost everywhere on (a, b) and 2) $\{\int_a^x f_n\}$ is an equi-absolutely continuous sequence on (a, b). It is shown that the limit of $\int_a^x f_n$ exists. $\int_a^x f$ is defined to be $\int_a^b f_n$. Some of the properties of this integral are shown to be the same as the corresponding properties of the Lebesgue integral (summable case).

11. *An interpretation of improper integrals*, by Mr. Hadi Haddad, University of Texas, introduced by the Secretary.

The usual extension of the definition of an integrable function from the Cauchy-Riemann point of view is made in terms of Lebesgue (summable) integrals for the case of unbounded range of the function and unbounded domain of definition. Illustrations are given.

12. *A set of equations for gas dynamics during combustion*, by Professor J. T. Hurt, Agricultural and Mechanical College of Texas, introduced by the Secretary.

From suitable averages of the properties of a gas during combustion, equations are derived very similar to those for an isentropic perfect gas without combustion.

13. *Can high school plane geometry and solid geometry be taught by means of analytic geometry?* by Professor H. J. Ettlinger, University of Texas.

An approach to Elementary Plane Geometry is proposed for a high school course in Euclidean Geometry. It is assumed that the linear number scale and the circular number scale are known. Distance along a ray and a "finite" set of angles from 0 to 180 is introduced. A point is an ordered pair of numbers (polar coordinates) and an angle is an ordered pair of rays with common end point. Parallel and perpendicular are defined. Line is defined. Distance is defined. Translation and Rotation are introduced. The usual plane geometry constructions are done with ruler and compasses. By means of projections on to a plane, solid geometry is developed.

C. R. SHERER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Marian College, Indianapolis, November 7, 1958. Valparaiso University, May 2, 1959.

IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.

KANSAS

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, Washington, D. C., December 6, 1958.

METROPOLITAN NEW YORK

MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.

MINNESOTA

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN, College of the Holy Cross, Worcester, Massachusetts, November 29, 1958.

NORTHERN CALIFORNIA, Stanford University, Stanford, January 17, 1959.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA, Lehigh University, Bethelehem, November 29, 1958.

ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, Spring, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State College, Tempe, Spring, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May 2, 1959.

ENGINEERS - SCIENTISTS

The Amherst Laboratory

Sylvania's

Center for Communications Research & Development

Mathematicians with advanced degrees and a lasting interest in some field of mathematical research will find in Sylvania's Amherst Laboratory both the opportunity and the environment to do creative work. Areas of interest include, BUT ARE NOT LIMITED TO, number theory, group theory, combinatorial design, stochastic processes, Monte Carlo techniques, information theory, game theory and statistical decision theory.

Scientists evaluating our employment opportunities will find that we amply reward creative ability, not only with a competitive, industrial-range salary, but also with liberal fringe benefits. Sylvania provides both life and hospitalization insurance as a company-paid benefit, and encourages membership in professional societies and subscriptions to technical publications by paying one-half the cost of dues and subscription fees.

The Amherst Laboratory is located in Western New York near Buffalo, Niagara Falls and the Province of Ontario.

Mathematicians interested in the Laboratory's expanding program of research are invited to write, in complete confidence, to the undersigned. A brief resume of education, experience and current interests, together with address and telephone number, will bring a prompt reply.

Dr. R. L. San Soucie
SYLVANIA ELECTRONIC SYSTEMS
A Division of
SYLVANIA
SYLVANIA ELECTRIC PRODUCTS INC.
1177 Wehrle Drive, Amherst 21, New York

★ *New and Recent Mathematical Texts*

INTRODUCTORY MATHEMATICAL ANALYSIS

Edgar D. Eaves and Robert L. Wilson

Written especially for students who have had only one year of high school algebra and plane geometry, this new text integrates the simple concepts of algebra, trigonometry, analytic geometry, and calculus into a unified course. Over 1200 problems and examples illustrate the meaning of new definitions and concepts as well as show how new operations are performed.

CALCULUS WITH ANALYTIC GEOMETRY

Richard E. Johnson and Fred Kiokemeister

A rigorous, yet intuitive, introduction to the calculus designed to fill textbook needs created by the modern trend toward comprehensive courses. Topics from analytic geometry are included to make the book self-contained for its geometrical applications. Limits are defined in the epsilon-delta manner, and all limit theorems are proved by this definition.

CALCULUS

Walter Leighton

Emphasizing fundamental concepts, this text attempts to avoid the extremes of rigor and non-rigor and provides the student with an overall insight into the nature of the calculus as well as a firm command of calculus techniques. Special stress is given to double integrals and curve tracing. Throughout, the book includes problems graded to challenge students of widely ranging abilities.

ARITHMETIC FOR COLLEGE STUDENTS

Edwin I. Stein

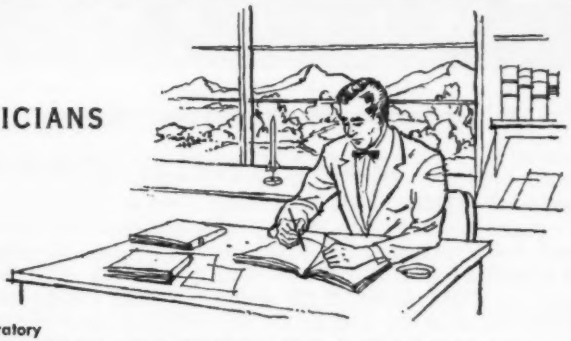
This newly published book is a complete basic text in arithmetic designed for refresher courses for college students and also for courses in teaching elementary arithmetic. The book includes thousands of graded examples in computation and numerous varied applications. As a special feature, three inventory tests are provided to diagnose general difficulties of the student.

For examination copies, write to **Arthur B. Conant:**

Allyn and Bacon • College Division

150 Tremont St., Boston 11, Mass.

MATHEMATICIANS



jpl

The Jet Propulsion Laboratory

now has opportunities open for Ph.D. Mathematicians of U.S. Citizenship to participate in many challenging problems relating to all phases of jet propulsion, aerodynamics, missile control and satellite flight.

Positions involve assignments in mathematical analysis, research in numerical analysis and consultation in applied mathematics. Computer facilities include an IBM-704 digital computer. Staff accommodations are excellent.

The laboratory is a continuing operation devoted to scientific research and development offering many opportunities for increasing responsibilities in an expanding activity.

JET PROPULSION LABORATORY
A DIVISION OF CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA • CALIFORNIA

---

3 reasons why

mathematicians everywhere repeatedly express their preference for

Rinehart Mathematical Tables, Formulas & Curves

by HAROLD LARSEN

Legibility

"One of the best sets of tables I have seen. The large page size and clear print make them easy to read." "A fine job—very attractively printed." "Splendid. I like the type and page arrangement."

Accuracy

"I am impressed with the degree of accuracy." "An uncommonly useful set of tables—accurate, well planned."

Inclusiveness

"The most complete coverage it has been my good fortune to see." "The selection of curves is excellent, more complete than usual." "I like the inclusion of important formulas."

Now in a handsome flexible binding for greater ease and convenience in desk use.

280 pages $2.50

Rinehart & COMPANY, INC. • 232 MADISON AVE., NEW YORK 16, N.Y.

WILEY BOOKS

New and forthcoming Wiley texts and references

SURVEYS IN APPLIED MATHEMATICS

Vol. I—ELASTICITY AND PLASTICITY. *By* J. N. Goodier *and* P. G. Hodge, Jr. *1958. 152 pages. $6.25.* **Vol. II—DYNAMICS AND NONLINEAR MECHANICS.** *By* E. Leimanis *and* N. Minorsky. *1958. 206 pages. $7.75.* **Vol. III—MATHEMATICAL ASPECTS OF SUBSONIC AND TRANSONIC GAS DYNAMICS.** *By* L. Bers. *1958. 164 pages. $7.75.* **Vol. IV—SOME ASPECTS OF ANALYSIS AND PROBABILITY.** *By* I. Kaplansky, E. Hewitt, M. Hall, Jr., *and* R. Fortet. *1958. 243 pages. $9.00.* **Vol. V—NUMERICAL ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS.** *By* G. E. Forsythe *and* P. C. Rosenbloom. *1958. 204 pages. $7.50.*

THE STRUCTURE OF ARITHMETIC AND ALGEBRA

By May Hickey Maria, *Brooklyn College.* Designed for the student with a limited background in science who wishes to understand the fundamental concepts that underlie the structure of algebra and arithmetic. It adopts a simple approach to general methods of modern mathematics and leisurely explores and develops the main properties of real numbers as logical consequences of a system of fundamental assumptions. *1958. 294 pages. $5.90.*

LECTURES ON ORDINARY DIFFERENTIAL EQUATIONS

By the late Witold Hurewicz, *formerly of Massachusetts Institute of Technology.* Based on the late author's lectures given at Brown University, this study offers students a rigorous exposition of the basic material of ordinary differential equations in the real domain. A Technology Press book, M.I.T. *1958. 122 pages. $5.00.*

INTRODUCTION TO FUNCTIONAL ANALYSIS

By Angus E. Taylor, *University of California, Los Angeles.* Treating functional analysis in a clear, direct manner, this text assists the student in learning and developing the fundamental ideas and theorems about linear spaces and linear operators. *1958. 423 pages. $12.50.*

STATISTICS: AN INTRODUCTION*

By D. A. S. Fraser, *University of Toronto.* Develops the basic mathematics of statistics from a prerequisite of calculus. It discusses recently developed techniques in estimation, hypothesis testing, confidence methods, and nonparametric methods. *Ready in December. Approx. 448 pages. Prob. $7.25.*

PLANNING OF EXPERIMENTS*

By D. R. Cox, *Birkbeck College, University of London.* Avoiding statistical and mathematical technicalities, this text presents in full the fundamental ideas of modern work on the statistical aspects of experimental design, with examples from a wide variety of topics. *1958. 308 pages. College edition $6.25.*

* *One of the Wiley Publications in Statistics, Walter A. Shewhart and S. S. Wilks, Editors.*

JOHN WILEY & SONS, Inc.

WILEY — BOOKS

INTRODUCING MATHEMATICS

By Floyd F. Helton, *Central College, Fayette, Mo.* Designed especially for freshman mathematics students who enter college with little or no mathematical background. Emphasis is on the *ideas* of elementary mathematics, together with enough manipulative routine to fix these ideas in practice. It develops in a careful, logical, and interesting manner the basic ideas and techniques of essential arithmetic, elements of algebra, and elements of geometry. *1958. 396 pages. $5.75.*

PRINCIPLES AND APPLICATIONS OF RANDOM NOISE THEORY

By Julius S. Bendat, *the Ramo-Wooldridge Corporation.* A thoroughly comprehensive explanation of the fundamental ideas of random noise theory. It contains a full treatment of Rice's representation of random noise, *plus* a development of the important Zero-Crossing problem. *1958. 431 pages. College edition $9.25.*

HANDBOOK OF AUTOMATION, COMPUTATION, AND CONTROL

Edited by Eugene M. Grabbe, Simon Ramo, *and* Dean E. Wooldridge, *all of the Ramo-Wooldridge Corporation.* The one complete guide to solving problems in all three fields. It is edited with a systems engineering emphasis in mind and is packed with data applicable to research, development, and design. Vol. I—CONTROL FUNDAMENTALS. *1958. 1020 pages. $17.00.* Vol. II—COMPUTERS AND DATA PROCESSING. *In press.* Vol. III—SYSTEMS AND COMPONENTS. *In press.*

SWITCHING CIRCUITS AND LOGICAL DESIGN

By Samuel Caldwell, *Massachusetts Institute of Technology.* Dealing with the methods used to handle the problems of circuit synthesis employing various kinds of components, this book clearly presents the principles of switching circuit design. *1958. 685 pages. College edition $11.75.*

INTRODUCTION TO DIFFERENCE EQUATIONS

With Illustrative Examples from Economics, Psychology, and Sociology

By Samuel Goldberg, *Oberlin College.* Offers the student a comprehensive introduction to the principles and applications of finite differences and difference equations. It includes numerous examples from the literature of economics, psychology, and sociology to demonstrate the various ways difference equations can be employed outside the field of mathematics. *1958. 260 pages. $6.75.*

FINITE QUEUING TABLES

By L. G. Peck *and* R. N. Hazelwood, *both of Arthur D. Little, Inc., Cambridge, Mass.* Publications in Operations Research No. 2. *1958. 208 pages. $8.50.*

READINGS IN LINEAR PROGRAMMING

By S. Vajda, *Royal Naval Scientific Service.* Describes the methods for determining how to employ a variety of items (trucks, machinery, etc.) in varying quantities to produce a desired result (reaching peak efficiency in shipping, production, etc.). *1958. 99 pages. $3.00.*

Send for examination copies.

440 Fourth Avenue, New York 16, N. Y.

FROM THE APPLETON-CENTURY MATHEMATICS SERIES

Analytic Geometry

By Edwin J. Purcell

Simply and clearly written, this book differs from other texts on the subject in its presentation of analytic geometry for its own sake, rather than as simply a preliminary to calculus, and in its emphasis on calculus as a logical system, not merely as a method in problem-solving.

289 pages, $4.50

College Geometry

By Leslie H. Miller

This important textbook for college courses in advanced Euclidean geometry stresses geometric constructions, with more emphasis than most texts on alternate methods of solution, extension of construction methods to general or to related problems, concluding with problems of a more advanced nature.

201 pages, $4.50

35 West 32nd Street — New York 1, New York

APPLETON-CENTURY-CROFTS, INC.

a new edition

Plane Trigonometry Second Edition
JOHN J. CORLISS & WINIFRED V. BERGLUND
397 pages 1958 $4.00

thorough treatment of business practices

The Mathematics of Finance
ROBERT CISSELL & HELEN CISSELL
Text: 198 pages 1956 $4.50
Tables: 88 pages

concise presentation

Analytic Geometry Second Edition
R. S. UNDERWOOD & FRED W. SPARKS
282 pages 1956 $3.75

a brief course

Practical Trigonometry
RALPH S. UNDERWOOD & HORACE S. WOODWARD
251 pages 1956 $3.25

HOUGHTON MIFFLIN COMPANY

Boston 7
New York 16
Chicago 16
Dallas 1
Palo Alto

HARPER

Announces

a modernized and revised edition of
the successful Peterson mathematics text

COLLEGE ALGEBRA

Second Edition

By THURMAN S. PETERSON, *Portland State College*

In Harper's Mathematics Series, under the
editorship of Charles A. Hutchinson

Many new topics are introduced that are currently in demand in college algebra courses. There is a clear-cut division at the end of Chapter 12 so that the first 12 chapters can be termed "intermediate algebra," and the last 11 may be used as the "college" course. Extensive chapter reviews appear at the ends of Chapter 6 and Chapter 12 to facilitate use at the more elementary level. New material on logic, number systems, calculus, matrices, partition theory, and sampling theory.—$4.00

And reminds you

of these other important Harper texts
in the same distinguished Series

By Thurman S. Peterson
for introductory and intermediate courses

ANALYTIC GEOMETRY AND CALCULUS—$6.00

ELEMENTS OF CALCULUS—$5.00

INTERMEDIATE ALGEBRA FOR COLLEGE STUDENTS—
Revised—$3.75

By Kenneth S. Miller (NYU)
for advanced courses

ADVANCED REAL CALCULUS—$5.00
ELEMENTS OF MODERN ABSTRACT ALGEBRA—$5.00

Harper & Brothers

49 East 33d Street, New York 16

For Second Semester Classes

INTRODUCTION TO MATHEMATICAL ANALYSIS WITH APPLICATONS TO PROBLEMS OF ECONOMICS

By Paul H. Daus, *University of California at Los Angeles, and* William M. Whyburn, *University of North Carolina*

This new textbook for a one-semester course in selected topics from analytic geometry and calculus is designed especially *for students of business and economics.* Recognizing the increasing need for these students to have a knowledge of mathematical topics and procedures which fall outside the scope of the usual courses in college algebra, trigonometry, and mathematics of finance, this text offers such essential grounding in the form of the fundamental principles of calculus and analytic geometry, *with applications in economics.* It might, advantageously, be studied concurrently with a first course in economics. The book was developed from a course offered in the School of Business Administration at the University of California at Los Angeles. The prerequisites necessary to study the book may be obtained from a course using any standard text in college algebra.

256 pp, 114 illus, 1958—$6.50

ORDINARY DIFFERENTIAL EQUATIONS

By Wilfred Kaplan, *University of Michigan*

The new ideas of instrumentation engineering are employed both as an illustration of the theory and as a way of better understanding the mathematics, in this textbook for a first course in differential equations. While it covers the topics usually included in an introductory course, it gives special emphasis to gaining a deeper understanding of the subject, rather than to learning special devices. To this end differential equations are related to their applications in physics and engineering, especially with regard to those aspects which are important for *servomechanisms, computers,* and *automatic controls.* This means that the book is especially suited for use by those going on in the field of instrumentation and control.

c. 480 pp, 150 illus, 1958—$8.50

ADVANCED CALCULUS

By Wilfred Kaplan

From a review in *The Mathematics Teacher:*

"This excellent book is an ideal blend of mathematical theory and mathematical practice. It is so filled with applications, examples from the sciences, and illustrative material that it might be thought to be primarily for the student of engineering, science, or applied mathematics; and yet the theory is undiluted and so carefully and concisely written as to meet the needs of any student."

679 pp, 241 illus, 1952—$9.00

ELEMENTARY DIFFERENTIAL EQUATIONS

By William Ted Martin *and* Eric Reissner, *Massachusetts Institute of Technology*

From a review in *Physics Today:*

"The authors have succeeded admirably in 'contributing not only to the reader's knowledge of differential equations but also to his ability to think about problems of mathematical analysis in general' and they have also provided both a good foundation and a stimulus for further interest in the deeper or more abstract problems of differential equations."

260 pp, 18 illus, 1956—$5.50

MATHEMATICAL ANALYSIS—A Modern Approach to Advanced Calculus

By Tom M. Apostol, *California Institute of Technology*

From a review in *Science:*

"The viewpoint of this textbook is excellently summarized by its subtitle . . . the material is serious substantial mathematics; topics traditionally from advanced calculus are supplemented with extra background for function theory. The style is unusually readable. The author displays a knack for breaking the development into proper sized bits for easy handling by the student."

553 pp, 88 illus, 1957—$9.00

Have your librarian write us for information about our Standing Order Inspection Plan

ADDISON-WESLEY PUBLISHING COMPANY, INC., Reading, Massachusetts, U.S.A.

Mathematics texts by Prentice-Hall . . .

ESSENTIAL MATHEMATICS FOR COLLEGE STUDENTS

by FRANCIS J. MUELLER, Maryland State Teachers College

An elementary text with a thorough, meaningful review of basic arithmetic processes. Algebra is developed as a natural extension of basic ideas of arithmetic. Topics through quadratics and imaginary numbers are covered. 1. An extensive number of examples of mathematical operations is provided to clarify text explanations, with step-by-step solutions for each. 2. In addition, approx. 1500 problems are included in the Exercises which follow important topics. Answers to these problems are supplied at the back of the book. 3. A total of 250 problems are given in Achievement Tests to serve as a review, in exercise form, of all the student has previously studied. 4. Tests and Exercises are printed on perforated right-hand pages for easy tear-out and submission.

238 pp. *Pub. 1958* *Paper bound* *Text price $3.95*

COLLEGE ALGEBRA, ALTERNATE EDITION

by MOSES RICHARDSON, Brooklyn College

Alternative edition of the successful College Algebra which has sold over 110,000 copies, includes new exercise lists throughout; unique new section on linear programming problems; rewritten chapter on probability; new section on non-decimal scales of notation for numbers; expanded section on inequalities. Contents include: The Number System of Algebra. What is Algebra? Algebraic Expressions. Functions and Graphs. Elementary Operations with Polynomials. Factoring of Polynomials; Elementary Operations with Fractional Expressions. Linear Equations and Linear Functions. Integral and Fractional Exponents. Radicals. Quadratic Equations and Quadratic Functions. Systems of Equations in Two Unknowns Involving Quadratics. Ratio, Proportion, and Variation. Complex Numbers. Theory of Equations. Determinants and Elimination Theory. Permutations and Combinations.

Approx. 544 pp. *Pub. 1958* *Text price $5.75*

PLANE TRIGONOMETRY, 3rd Edition

by FRED W. SPARKS, Texas Technological College and PAUL K. REES, Louisiana State University

A highly readable and teachable text which is adaptable to students of varying degrees of ability. It includes all the basic ideas needed for engineering courses and for later courses in mathematics. A good balance between theory and practice, and between analytical and numerical work, allows you to adapt the text material to your own classes. The exercises are right after discussions and examples to which they relate, so that the material for assignments is unified in one place. From 40 to 45 assignments may be given from the natural order of the contents.

275 pp. with tables *Pub. 1952* *With Tables, Text price $4.75*
199 pp. without tables *Without Tables, Text price $4.25*

To receive approval copies promptly, write: Box 903

PRENTICE-HALL, Inc.

Englewood Cliffs, New Jersey

outstanding texts by WILLIAM L. HART

Analytic Geometry and Calculus—
The first third of the book includes basic analytic geometry, part of differential calculus, and a substantial chapter on integral calculus restricted to algebraic functions, with a few applications of definite integrals and an introduction to differential equations. Integration is introduced early, and the definite integral is defined and used numerically to a moderate extent before the introduction of indefinite integration. The text maintains a sensible balance between theory and training in fundamental techniques, with ample provision for superior students. *648p. text* *$7.00*

Trigonometry is a text designed for students who will use trigonometry in more advanced mathematics. It offers the full content of plane trigonometry and includes spherical trigonometry. The well-rounded discussion of logarithmic computation and numerical trigonometry is amplified by a wealth of applications. Analytic trigonometry is emphasized, in view of its importance in later collegiate mathematics. Hundreds of illustrative examples and extensive problem material . . . unusually complete tables. *368p. (211p. text)* *$4.25*

D. C. HEATH AND COMPANY

THE THEORY OF GROUPS

by Marshall Hall, Jr., The Ohio State University

This text presents the fundamentals of the theory of groups. The first ten chapters, constituting a text for a course in the theory of groups, presupposes only an elementary knowledge of modern algebra. The last ten chapters, providing a wide selection from the most recent and active areas of research, may be used in course work or read individually for reference.

Coming January 1959

ELEMENTARY MATRIX ALGEBRA

by Franz E. Hohn, University of Illinois

This extremely teachable text presents those aspects of elementary matrix algebra which are most commonly applied in the physical and social sciences. Unlike many abstract mathematical books, this text treats fully the fundamental properties of determinants. Devoid of unnecessary mathematical formalism, the book contains illustrative worked examples and a variety of exercises for the student.

Published November 1958

INTRODUCTION TO PROBABILITY AND STATISTICS

by B. W. Lindgren and G. W. McElrath, University of Minnesota

Assuming a background in calculus, this text is designed for engineering students. It provides many classical and modern statistical methods based on a preliminary, thorough treatment of probability theory. Numerous problems and illustrations, as well as such modern techniques as non-parametric tests, are included. The last chapter contains a short introduction to fields for further study.

Coming Spring 1959

TABLES OF INTEGRALS AND OTHER MATHEMATICAL DATA, *Third Edition*

by Herbert B. Dwight, Professor Emeritus, Massachusetts Institute of Technology

A standard reference for mathematics courses, this book has been revised to include 38 new pages of tables—including sine and cosine of hundredths of degrees—making it a most complete and useful tool for mathematicians.

1957, 288 pages, $5.00

The Macmillan Company

60 FIFTH AVENUE, NEW YORK 11, N. Y.

Important McGraw-Hill Books

MATHEMATICAL METHODS OF OPERATIONS RESEARCH

By **THOMAS L. SAATY,** *Office of Naval Research. Ready in January*

The first graduate level text in the field. Covers the principal mathematical methods, and brings together the previously scattered theoretical and illustrative literature. The first part of the book which provides an initial orientation toward operations research: historically, scientifically, and mathematically and logically, constitutes a well-integrated framework for working on operations research projects. Examples and ideas on most of the useful classical methods in forming models, the subject of optimazation, the main conceptual points of game theory, applications of probability, and a unique resume on queuing theory follow.

DIFFERENTIAL AND INTEGRAL CALCULUS

By **HAROLD MAILE BACON,** *Stanford University. 559 pages, $6.50*

A highly readable and careful treatment of the theory of calculus illustrated by many examples and exercises in varying degrees of complexity. A clear understanding of the calculus as well as a practical working knowledge of the subject is presented. Starting with a differentiation of all types of functions of a single independent variable, a treatment of the indefinite integral with methods of formal integration follows. The definite integral precedes a consideration of the functions of more than one independent variable.

PLANE TRIGONOMETRY

By **E. R. HEINEMAN,** *Texas Technological College. Second Edition. 256 pages—With Tables, $3.95. Text alone, $3.50*

This exceptionally well-organized and teachable text, slanted toward the needs of students who will continue mathematics through calculus, promotes to an unusual degree the student's ability to think independently and logically. Memory work is reduced to a minimum and all unnecessary formulas and concepts are omitted. Although it remains a traditional treatment of the subject, with a standard approach to triangle solving, the function concepts receive more emphasis. The analytic aspects of trigonometry are stressed; complete coverage is given to the solution of triangles; and definite instructions are given for proving identities and solving trigonometric equations. A careful explanation of approximations and significant figures is given early in the text, and the new characteristic rule for logarithms is presented. All problem sets are carefully graded.

Also available . . . **PLANE TRIGONOMETRY**—*Alternate Edition. 252 pages—With Tables, $3.95. Text alone, $3.50.* Contains a new set of problems and exercises, but is otherwise the same as the first edition of the regular text.

SEND FOR COPIES ON APPROVAL

* **McGRAW-HILL BOOK COMPANY, INC.** *

330 WEST 42nd STREET . . . NEW YORK 36, N. Y.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.